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Messenger

THE

MESSENGER OF MATHEMATICS,

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MESSENGER OF MATHEMATICS.

ON A RULE FOR ABBREVIATING THE CALCULATION OF THE NUMBER OF IN- OR CO-VARIANTS OF A GIVEN ORDER AND WEIGHT IN THE COEFFICIENTS OF A BINARY QUANTIC OF A GIVEN DEGREE.

By Professor *J. J. Sylvester.*

If i is the degree of a quantic we know now by *apodictic* reasoning that the number of its in- or co-variants of order j and of weight w in the coefficients is $(w : i, j) - \{(w-1) : i, j\}$, where in general $(x : i, j)$ denotes the number of modes of composing x with j numbers each having any value from 0 to i (both inclusive) or (what is the same thing) with i numbers each having any value from 0 to j. The object of this note is to show how to calculate the *difference* between the two denumerants above given without calculating each of them separately, whereby the actual amount of calculation required will be reduced to a small fraction of what it would otherwise be. I shall not stop to draw theoretical consequences from this theorem, but present it to the readers of the *Messenger* in the way it has occurred to me as a rule for abbreviating labor.

It is founded on the exhaustive method of representing partition systems by following a dictionary order of sequence, and will be best understood by beginning with an example.

Suppose then that $w=7$, $i=5, j=4$, we may find $(7 : 5, 4)$ by setting out and counting the arrangements where 4 is the number of parts and 5 the limit to each part, viz. 5.2, 5.1.1, 4.3, 4.2.1, 4.1.1.1, 3.3.1, 3.2.2, 3.2.1.1, 2.2.2.1.

For brevity the zeros required to fill up the number of parts to 4 are omitted in this table.

To find $(6 : 5, 4)$ we may consider

1° Those arrangements which begin with 5.

2° Those arrangements which begin with a number less than 5.

To obtain the latter also arranged in dictionary order of sequence, we may (subject to an exception to be stated immediately below) proceed by diminishing each initial number in the above table by unity.

The exception to be made is where 2 initial numbers are alike, as in 3.3.1; 2.2.2.1. These arrangements must not be counted in as the arrangements 2.3.1, 1.2.2.1 will already have been obtained from 4.2.1, 3.2.1.1 respectively.

Hence the number of arrangements in the above table to be preserved is less by 2 than the total number.

On the other hand we shall have the arrangement 5.1, to which there is nothing corresponding in the table for (7 : 5, 4). Hence the difference required is

$$2-1, \textit{i.e.}\ (7:5,4)-(6:5,4)=1.$$

Let us take as a second example w (the weight) 12, i (the limit to each part) 6, and j (the number of parts) 4.

Let A be the table for (12 : 6, 4) in dictionary order, and let A' be the part of the table for (11 : 6, 4), also arranged in dictionary order, for which 6 is nowhere the initial term. Let A_1 be what A becomes when each initial number is diminished by unity.

Then, by the same reasoning as above, we must have $A'-A_1=6.6,\ 5.5.2,\ 5.5.1.1,\ 4.4.4,\ 4.4.3.1,\ 4.4.2.2,\ 3.3.3.3,$ 7 in number.

Also calling B the part of the table for (11 : 6, 4), beginning with 6 we have $B=6.5,\ 6.4.1,\ 6.3.2,\ 6.3.1.1,\ 6.2.2.1,\ 5$ in number.

Hence (12 : 6, 4) − (11 : 6, 4) = 7 − 5, = 2.

To verify this, let us interchange the values 6 and 4, this by a well-known theorem leaves the value of each denumerant unaltered.

We have now $A'-A_1=4.4.4,\ 4.4.3.1,\ 4.4.2.2,\ 4.4.2.1.1,$ 4.4.1.1.1.1, 3.3.3.3, 3.3.3.2.1, 3.3.3.1.1.1, 3.3.2.2.2, 3.3.2.2.1.1, 2.2.2.2.2.2, number is 11.

Also $B=4.4.3,\ 4.4.2.1,\ 4.4.1.1.1,\ 4.3.3.1,\ 4.3.2.2,\ 4.3.2.1.1,$ 4.3.1.1.1.1, 4.2.2.2.1, 4.2.2.1.1.1, number is 9, and thus (12 : 4, 6) − (11 : 4, 6) = 11 − 9 = 2 as before. Evidently this identity between the two forms of $(w : i, 5)-\{(w-1) : i, 5\}$ given by this method and also the incapability of this difference becoming negative when w is not greater than $\frac{1}{2}ij$, which I have elsewhere demonstrated, may be made to yield arithmetical properties of a new kind, and not unlikely to prove very valuable in certain parts of the theory of numbers; but what has impressed itself on my mind is the *enormous saving* of

labor in the actual business of calculating invariantive formulæ, which this method confers. The existence of a perfectly definite table exhibiting an exhaustive arrangement of *ruled partitions* (as I call partitions subject to the two indices i, j) in itself constitutes a theorem (however simple), and the method above given is a further and more recondite theorem deduced from it, combined of course with other *intuitional* propositions.

Let us take as another example $w = 20$, $i = 13$, $j = 3$.

Here $A' - A_1 = 10.10,\ 9.9.2,\ 8.8.4,\ 7.7.6$. $B = 13.6,\ 13.5.1,\ 13.4.2,\ 13.3.3$. Therefore $(20 : 13, 3) - (19 : 13, 3) = 0$.

Again let us calculate $(40 : 20, 4) - (39 : 20 : 4)$.

Here $A' - A_1 = 20.20,\ 19.19.2,\ 19.19.1.1,\ 18.18.4,\ 18.18.3.1,\ 18.18.2.2,\ 17.17.6,\ 17.17.5.1,\ 17.17.4.2,\ 17.17.3.3$, and similarly 16.16 with 5 duads, 15.15 with 6 duads, 14.14 with 7 duads. Also 13.13 with 13.1, 12.2, 11.3, 10.4, 9.5, 8.6, 7.7, 12.12 with 12.4, 11.5, 10.6, 9.7, 8.8, 11.11 with 11.7, 10.8, 9.9 10.10.10.10. Thus the number of terms in $A' - A_1$ is

$$(1 + 2 + 3 + 4 + 5 + 6 + 7) + (7 + 5 + 3 + 1) = 44.$$

And B is composed of arrangements containing 20, together with the number of triads into which $39 - 20$, *i.e.* 19 can be decomposed, none greater than 20, *i.e.* the number of terms in B is $19 : 20, 3$, which is the same as the absolute number of modes of resolving 19 into 3 parts or fewer, which is

$$1 + 1 + 2 + 2 + 3 + 3 + 4 + 4 + (10 : 9, 2) + (9 : 10, 2) + (8 : 11, 2) + (7 : 12, 2) = 25 + 5 + 5 + 3 + 2 = 40.$$

Thus $(40 : 20, 4) - (39 : 20, 4) = 44 - 40 = 4$,

which is easily verified, for the difference between the above two denumerants is the number of linearly independent invariants of the 20th order to a quartic, *i.e.* is the number of ways of composing 20 with 2 and 3 (the orders of the fundamental invariants) which is 4 as found above.

The method thus simply and almost intuitively deduced, may be expressed in the form of a theorem as follows:

$$\Sigma_{q=0}^{q=i}(w - 2q : q, j - 2) - (w - i - 1 : i, j - 1) = (w : i, j) - (w - 1 : i, j) = \Sigma_{q=0}^{q=j}(w - 2q : q, i - 2) - (w - j - 1 : j, i - 1).$$

The inferior unit is taken zero for the purpose of theoretical simplicity. Let the effective value of this limit be called $[q]$, and consider the first of the above three equals.

The value of $[q]$ is given by the condition that

$$w - 2[q] \text{ shall be not greater than } (j-2)[q],$$

i.e. $$[q] \text{ not less than } \frac{w}{j},$$

i.e. $[q]$ is $\frac{w}{j}$ if $\frac{w}{j}$ is an integer, $\frac{w}{j}+1$ if $\frac{w}{j}$ is fractional,

i.e. $$[q] = E\frac{w+j-1}{j};$$

(E standing as usual for the integer part of the quantity which it precedes).

The number of actual terms differing from zero under the sign of summation is therefore

$$i+1-E\frac{w+j-1}{j}, \quad i.e.\ 1+E\frac{ij-w}{j},$$

similarly the number of terms under the sign of summation in the conjugate form will be $1+E\frac{ij-w}{i}$.

Thus the first or second expression will be the best to employ, according as j is greater or less than i.

Again, since $(w : i, j) = (ij - w : i, j)$,

we may in place of

$$(w : i, j) - (w-1 : i, j),$$

employ $$(w' : i, j) - (w'+1 : i, j),$$

which is $$-[(w'+1 : i, j) - (w' : i, j)].$$

Hence, we may always secure in the application of this method, that the numerator in $E\frac{ij-w}{i}$ or in $E\frac{ij-w}{j}$ shall not be greater than $\frac{1}{2}ij$. Supposing j to be greater or not less than i, so that the first formula is applied, it will be found most convenient, so long as q is less or not greater than $j-2$, to consider q the number of the parts in any of the quantities $(w-2q : q, j-2)$, and $j-2$ the limit to the magnitude of each part, and until q becomes equal to $i-1$, this hypothesis will always be the case. When $q=i$ or when $q=i$ and $q=i-1$ in the respective cases of j being only one unit greater than i or equal to i, the two indices $q : j-2$ may with advantage be reversed. For any other values of $j-i$

the order of the indices need not be disturbed. It may be worth while to call attention to the *two* independent theorems of reciprocity made use of in the preceding discussion, indicated by the equations

$$\begin{aligned}&(w : i, j)\\ &= (w : j, i)\\ &= (ij - w : i, j)\\ &= (ij - w : j, i),\end{aligned}$$

both of them of importance in the theory of invariants after the English method.

Johns Hopkins University, Baltimore,
November 26, 1877.

Addition.

Notwithstanding what has been stated above as to the choice between the two formulæ representing $\Delta(w : i, j)$, the advantage of diminishing the smaller of the two indices i, j, will simplify the calculations to a degree that far more than outweighs the disadvantage of increasing the number of terms under the sign of summation. Let us suppose then that j is less than w, and that $\Delta(w : i, j)$ is positive, representing in fact indifferently the number of linearly independent covariants of order i to a quantic of degree j, or of order j to a quantic of degree i. Then, unless these covariants are invariants, we must have $w < \frac{1}{2}ij$.

Consequently, the best formula to apply in such case will be obtained by writing $\Delta(w : i, j)$

$$\begin{aligned}&= (ij - w : i, j) - (ij - w + 1 : i, j)\\ &= -\Delta(ij - w + 1 : i, j)\\ &= (ij - i - w : i, j - 1) - \Sigma_{q=0}^{q=i}(ij - w + 1 - 2q : q, j - 2).\end{aligned}$$

The number of terms other than zero under the sign of summation will then be $1 + E\frac{w}{j}$.

For the case of invariants we may with at least equal advantage use the formula

$$\Sigma_{q=0}^{q=i}(\tfrac{1}{2}ij - 2q : q, j - 2) - (\tfrac{1}{2}ij - 1 - i : i, j - 1).$$

Let us apply this to the case of finding

$$\Delta\left(\frac{18.5}{2} : 18, 5\right), \text{ i.e. } (45 : 18, 5).$$

In the work below I use, whenever useful, the formula of transformation $(x : i, j) = (ij - x : i, j)$, and employ $\frac{\mu}{3}$ to denote the number of ways of breaking up μ into three or fewer parts, which we know is the nearest integer to $\frac{(\mu+3)^2}{12}$; and in like manner $\frac{\nu}{2}$ for the number of ways of breaking up ν into two parts: also in place of $(x : k, 3)$, whenever k is at least as great as x, I use the obviously equivalent value $\frac{x}{3}$.

Let us then first calculate

$$\Sigma_{q=9}^{q=18}[45 - 2q : q, 3], \text{ say } S.$$

The values of q inferior to 9 will give quantities in which $3q < 45 - 2q$, and which will therefore be zero.

We have thus

$$\begin{aligned} S = {} & (9 : 18, 3) + (11 : 17, 3) + (13 : 16, 3) + (15 : 15, 3) \\ & + (17 : 14, 3) + (19 : 13, 3) + (21 : 12, 3) + (23 : 11, 3) \\ & + (25 : 10, 3) + (27 : 9, 3) \\ = {} & \tfrac{9}{3} + \tfrac{11}{3} + \tfrac{13}{3} + \tfrac{15}{3} + (17 : 14, 3) + (19 : 13, 3) + (15 : 12, 3) \\ & + (10 : 11, 3) + (5 : 10, 3) + (0 : 9, 3). \end{aligned}$$

Also

$$(17 : 14, 3) = (17 : 17, 3) - \tfrac{1}{2} - \tfrac{2}{2} - \tfrac{3}{2} = \tfrac{17}{3} - 1 - 2 - 2 = \tfrac{17}{3} - 5,$$

$$(19 : 13, 3) = (19 : 19, 3) - \tfrac{1}{2} - \tfrac{2}{2} - \tfrac{3}{2} - \tfrac{4}{2} - \tfrac{5}{2} - \tfrac{6}{2} = (19 : 19, 3) - 15,$$

$$(15 : 12, 3) = (15 : 15, 3) - \tfrac{1}{2} - \tfrac{2}{2} - \tfrac{3}{2} = \tfrac{15}{3} - 5.$$

Thus $$\begin{aligned} S = {} & \tfrac{9}{3} + \tfrac{11}{3} + \tfrac{13}{3} + \tfrac{15}{3} + \tfrac{17}{3} + \tfrac{19}{3} + \tfrac{15}{3} + \tfrac{10}{3} + \tfrac{5}{3} + \tfrac{0}{3} - 25 \\ = {} & \tfrac{9}{3} + \tfrac{5}{3} + \tfrac{8}{3} + \tfrac{10}{3} + \tfrac{11}{3} + \tfrac{13}{3} + 2.\tfrac{15}{3} + \tfrac{17}{3} + \tfrac{19}{3} - 25. \end{aligned}$$

Again let $S' = (44 - 18 : 18, 4) = (26 : 18, 4)$.

Then

$$\begin{aligned} S' = {} & (8 : 18, 3) + (9 : 17, 3) + (10 : 16, 3) + (11 : 15, 3) \\ & + (12 : 14, 3) + (13 : 13, 3) + (14 : 12, 3) + (15 : 11, 3) \\ & + (16 : 10, 3) + (17 : 9, 3) + (18 : 8, 3) + (19 : 7, 3) \\ = {} & \tfrac{8}{3} + \tfrac{8}{3} + \tfrac{10}{3} + \tfrac{11}{3} + \tfrac{12}{3} + \tfrac{13}{3} + (\tfrac{14}{3} - 3) + (\tfrac{15}{3} - 8) \\ & + (\tfrac{14}{3} - 8) + (\tfrac{10}{3} - 1) + \tfrac{6}{3} + \tfrac{7}{3} - 20 \\ = {} & \tfrac{7}{3} + \tfrac{6}{3} + \tfrac{8}{3} + \tfrac{9}{3} + 2.\tfrac{10}{3} + \tfrac{11}{3} + \tfrac{12}{3} + \tfrac{13}{3} + 2.\tfrac{14}{3} + \tfrac{15}{3} - 20; \end{aligned}$$

therefore

$$S - S' = \tfrac{0}{3} - \tfrac{2}{3} + \tfrac{5}{3} - \tfrac{8}{3} - \tfrac{8}{3} - \tfrac{10}{3} - \tfrac{12}{3} - 2.\,\tfrac{14}{3} + \tfrac{15}{3} + \tfrac{17}{3} + \tfrac{19}{3} - 5$$
$$= 1 - 2 + 5 - 7 - 10 - 14 - 19 - 48 + 27 + 33 + 40 - 5$$
$$= 106 - 105 = 1,$$

which is right, there being just one invariant to the quantic of the eighteenth order in the coefficients, so that $\Delta(45 : 18, 5) = 1$.

It appears from the tables given in M. Faà de Bruno's valuable *Théorie des Formes Binaires*, Turin, 1877, that this invariant contains 848 terms. Therefore the value of $(18 : 8, 5)$ is very considerably greater* than 848.

Thus, by the direct method of calculating $\Delta(45 : 18, 5)$, many more than 1695 terms would have required setting out.

There is one case which deserves special consideration, viz., when one of the indices i or j becomes infinite.

The function $\Delta(w : \mu, \infty)$ then represents the total number of in- and co-variants of weight w of any given order not less than w to a quantic of the μ^{th} degree.

The two formulæ for this case become respectively

$$\Sigma_{q=0}^{q=\infty} [w - 2q : q, \mu],$$

and
$$\Sigma_{q=0}^{q=\mu} [w - 2q : q, \infty] - [w - \mu - 1 : \mu, \infty],$$

or if we agree to understand in all cases by $\frac{n}{m}$ the number of ways of making up n with the integers $0, 1, 2, 3 \ldots m$, or, what is the same, the number of ways of breaking up n into m or fewer parts, the second formula becomes

$$\Sigma_{q=0}^{q=\mu} \frac{w - 2q}{q} - \frac{w - \mu - 1}{\mu};$$

of these two the first is by far the most expeditious.

Let us take as an example $\Delta(20 : 6, \infty)$, *i.e.* $\frac{20}{6} - \frac{19}{6}$.

The first formula (neglecting the values of q which make $w - 2q$ negative and those which make $4q < (w - 2q)$, will give for the value of Δ

$$\begin{aligned} (0 : 10, 4) &= (0) \\ +(2 : 9, 4) &\quad +(2) \\ +(4 : 8, 4) &\quad +(4) \end{aligned}$$

---

* I say very considerably greater than, because only a certain number of the terms which satisfy the required conditions of order and weight actually appear in the octodecimal invariant in question. Thus *ex. gr.* there is no f^9, no f^8, and of the (10, 11, 5) *i.e.* $\frac{10}{6}$ terms which might contain f^7, only six, viz. the terms contained in $a(ac - b^2)^5$ actually make their appearance in it.

$$\begin{array}{ll} +(6:7,4) & +\frac{6}{4} \\ +(8:6,4) & +(8:6,4) \\ +(10:5,4) & +(10:5,4) \\ +(12:4,4) & +(4:4,4),\ i.e.\ (4), \end{array}$$

where in general (m) means *all* the modes of breaking up m into parts. The value of $(10:5,4)$ will be easily found to be 9 of $(8:6,4)$ 12 and of $\frac{6}{4}$, 9, also of (4) is 5. The value of $\frac{20}{6} - \frac{10}{6}$ thus becomes $1+2+5+9+12+9+5=43$.

By the second formula the value of the same quantity would be $\frac{8}{6} + \frac{10}{5} + \frac{12}{4} + \frac{14}{3} + \frac{16}{2} + \frac{18}{1} - \frac{13}{5}$, which would be exceedingly tedious to calculate.

In like manner if w is odd we shall have a series of denumerants of the form

$$\left(1:\frac{w-1}{2},\ \mu\right),\ \left(3:\frac{w-3}{2},\ \mu\right),\ \left(5:\frac{w-5}{2},\ \mu\right),\ \&c.$$

Thus *ex. qr.* $\frac{11}{6} - \frac{10}{6}$ (*i.e.* the number of in- and co-variants to a sextic of weight 11 and of any given order not inferior to 11, or, if we please to vary the expression, the number of in- and co-variants of weight 11 and the sixth order to any quantic of a degree not inferior to 11) will be

$$\left.\begin{array}{l} (1:5,4) \\ +(3:4,4) \\ +(5:3,4) \\ +(7:2,4) \end{array}\right\} = \left\{\begin{array}{l} (1) \\ +(3) \\ +(5:3,4) \\ +(1:2,4)\ i.e.\ (1) \end{array}\right.$$

$$= 1+3+4+1=9.$$

November 28, 1877.

ON THE OCCURRENCE OF THE HIGHER TRANSCENDENTS IN CERTAIN MECHANICAL PROBLEMS.

By *W. H. L. Russell, F.R.S.*

(Continued from vol. VII., p. 139).

(7) A CYLINDER is placed with its axis horizontal upon a smooth inclined plane, a string one end of which is attached to a fixed point is wound round the cylinder; determine the motion.

When the distance of the fixed point from the plane is equal to the radius of the cylinder, the problem is much simplified. Under this condition it has been discussed by

Euler, whose solution will be found in Walton's *Mechanical Problems*. I here investigate the general case which is much more complicated, and I think possesses considerable interest.

Let A be the fixed point, K the centre of a circular section made by plane through A perpendicular to the axis of the cylinder; C the point where this section touches the inclined plane; let the plane of the circular section be the plane of (xy), A the origin AX, AY, the axes respectively parallel and perpendicular to the inclined plane.

Let E be the point in which the string from A meets the cylinder, D the point where the axis of (y) meets the inclined plane. Also let KF be a line drawn from K perpendicular to the axis of (y) and meeting it in F, G the intersection of AE and KF.

Let $AM=x$, $MK=y$ be the coordinates of K, also let KC radius of cylinder $=a$, $AF=c$, then we shall find

$$AG=\frac{acx}{a^2-c^2}-\frac{c^2\sqrt{(x^2+c^2-a^2)}}{a^2-c^2},$$

$$EG=\frac{a^2\sqrt{(x^2+c^2-a^2)}}{a^2-c^2}-\frac{cax}{a^2-c^2},$$

$$KG=\frac{a^2x}{a^2-c^2}-\frac{ac\sqrt{(x^2+c^2-a^2)}}{a^2-c^2},$$

$$FG=\frac{ac\sqrt{(x^2+c^2-a^2)}}{a^2-c^2}-\frac{c^2x}{a^2-c^2}.$$

Hence also we shall have

$$\cos KGE=\frac{a\sqrt{(x^2+c^2-a^2)}-cx}{ax-c\sqrt{(x^2+c^2-a^2)}}.$$

Consequently, adopting the usual notation, the equations of motion are

$$m\frac{d^2x}{dt^2}=mg\cos\alpha-T\frac{a\sqrt{(x^2+c^2-a^2)}-cx}{ax-c\sqrt{(x^2+c^2-a^2)}},$$

$$mk^2\frac{d^2\theta}{dt^2}=Ta,$$

when α is the angle of inclination of inclined plane; there is no equation in y since $y=c$. The equation of *vis viva* manifestly implies that

$$a\frac{d\theta}{dx}=\frac{a\sqrt{(x^2+c^2-a^2)}-cx}{ax-c\sqrt{(x^2+c^2-a^2)}},$$

and the interest of the problem mainly consists in proving this equation by geometry, which may be effected as follows:

Let the centre of the sphere move through a small space from K to K', and the point of contact of string and cylinder from E to E', yet so that the length of string from fixed point to point of contact may remain the same, in other words, that $AE = AE'$. This implies that E' is the commencement, E the termination of the first element of contact in the two cases. Draw EKD to meet the axis of (x) in D, $E'K'D'$ to meet the axis of (x) in D'; KM, $K'M'$ perpendicular to the axis. Then $EK = E'K' = a$, $KM = K'M' = c$; $d\theta$ is the angle between ED and $E'D'$, also dx is represented by MM'.

Then, since $K'A^2 = (x + dx)^2 + c^2$, we have, retaining the first order of infinitesimals,

$$\cos K'AM = \frac{M'A}{K'A} = \frac{x}{\sqrt{(x^2 + c^2)}} + \frac{c^2 dx}{(x^2 + c^2)^{\frac{3}{2}}},$$

and, similarly,

$$\sin K'AM = \frac{c}{\sqrt{(x^2 + c^2)}} - \frac{cx\,dx}{(x^2 + c^2)^{\frac{3}{2}}},$$

$$\cos E'K'A = \frac{a}{\sqrt{(x^2 + c^2)}} + \frac{x\,dx}{a} \cdot \frac{x^2 + c^2 - a^2}{(x^2 + c^2)^{\frac{3}{2}}},$$

whence $$\sin E'K'A = \frac{\sqrt{(x^2 + c^2 - a^2)}}{\sqrt{(x^2 + c^2)}} - \frac{x\,dx\,\sqrt{(x^2 + c^2 - a^2)}}{(x^2 + c^2)^{\frac{3}{2}}},$$

also $$\sin EDA = \frac{x\sqrt{(x^2 + c^2 - a^2)} - ac}{x^2 + c^2},$$

$$\cos E'D'A' = \cos K'AM \cos E'K'A + \sin K'AM \sin E'K'A$$

$$= \frac{ax + c\sqrt{(x^2 + c^2 - a^2)}}{x^2 + c^2} + \frac{1}{a}\left\{\frac{ac - x\sqrt{(x^2 + c^2 - a^2)}}{x^2 + c^2}\right\}^2 dx$$

where $$\cos EDA = \frac{ax + c\sqrt{(x^2 + c^2 - a^2)}}{x^2 + c^2}.$$

Hence subtracting these expressions

$$2 \sin \tfrac{1}{2}(D + D') \sin \tfrac{1}{2}(D - D') = \frac{1}{a}\left\{\frac{ac - x\sqrt{(x^2 + c^2 - a^2)}}{x^2 + c^2}\right\}^2 dx.$$

But $\tfrac{1}{2}(D + D') = EDA$, $\tfrac{1}{2}(D - D') = \tfrac{1}{2}(d\theta)$;

and, therefore,

$$a\frac{d\theta}{dx}=\frac{x\sqrt{(x^2+c^2-a^2)}-ac}{x^2+c^2}=\frac{a\sqrt{(x^2+c^2-a^2)}-cx}{ax-c\sqrt{(x^2+c^2-a^2)}},$$

as is immediately proved by multiplication.

Hence, the equation of *vis viva* is demonstrated geometrically, and the solution of the problem is given by an equation of the form

$$t=\int dx\sqrt{[\phi(x)+\sqrt{\{\chi(x)\}}]},$$

when $\phi(x)$ and $\chi(x)$ are rational functions of (x).

By means of an integral of the same form we may determine the motion of an elliptic cylinder rolling on a rough horizontal plane, of an elliptic cylinder gliding in contact with two smooth planes, one horizontal the other vertical, and of a sphere rolling down a rough elliptic cylinder.

We arrive at the general conclusion that an immense number of dynamical problems, when the motion is in parallel planes, can be resolved by means of hyperelliptic integrals, and the higher algebraical transcendents; and, moreover, that a number of curious geometrical theorems are naturally suggested by the application of the principle of *vis viva* to the equations of motion in complicated cases.

NOTE ON THE THEOREM IN KINEMATICS (VOL. VII., p. 125).

By *C. Leudesdorf, M.A.*

AFTER sending this result to the *Messenger*, a shorter proof than the one I had given was suggested to me by Mr. Elliott's paper, published in the February number. It seems even simpler than that given by Mr. Kempe at p. 166. If the coordinates of A, B, C, P be (p_1q_1), (p_2q_2), (p_3q_3), (pq) respectively, it is seen without difficulty that

$$p=xp_1+yp_2+zp_3,\quad q=xq_1+yq_2+zq_3,$$

and that

$$\begin{aligned}qdp&=(xq_1+yq_2+zq_3)(xdp_1+ydp_2+zdp_3)\\&=xq_1dp_1+yq_2dp_2+zq_3dp_3-yz(q_2-q_3)(dp_2-dp_3)\\&\qquad-zx(q_3-q_1)(dp_3-dp_1)-xy(q_1-q_2)(dp_1-dp_2),\end{aligned}$$

so that, integrating over a complete circuit,

$$(P)=x(A)+y(B)+z(C)-yz\int(q_2-q_3)\,d(p_2-p_3)-\ldots-\ldots.$$

But $\int(q_2-q_3)\,d(p_2-p_3)$ is the area traced out by C

relatively to B, which, since $BC=a$ always, is equal to $n\pi a^2$ if the system make n complete rotations, and to zero if the system make a partial rotation and a return, as described by Mr. Kempe. Similarly for the other two integrals; so that

$$(P)-x(A)-y(B)-z(C)=-n\pi(a^2yz+b^2zx+c^2xy) \text{ or } 0$$
$$= n\pi t^2 \text{ or } 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

t being the length of the tangent from P to the circle round ABC.

The formula may also be proved by supposing the motion of the four points to be produced by their being carried with a closed curve which rolls on another in its plane. It is not necessary to give this proof, but its form has suggested to me the existence of a theorem of a form analogous to (1), connecting the areas of the pedals of any closed curve with regard to four points A, B, C, P in its plane taken successively as origin. Denoting these areas by (A), (B), (C), (P), that of the closed curve by (S), and the angle made with a fixed direction by the tangent to this curve by ϕ, we have

$$2(P)=(S)+\tfrac{1}{2}\int OP^2\,d\phi,$$

and three similar equations, O being any point in the plane. Thus

$$(P)-x(A)-y(B)-z(C)$$
$$=\tfrac{1}{4}\int(OP^2-xOA^2-yOB^2-zOC^2)\,d\phi$$
$$=\tfrac{1}{4}\int t^2 d\phi,$$

(by an easily proved geometrical relation)

$$=\tfrac{1}{2}n\pi t^2 \text{ or } 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

according as the tangent turns through $2n\pi$, or through any angle and back again to its initial position, as its point of contact moves round the curve. If P lie on AB, dividing it in the ratio $c:c'$,

$$(P)-\frac{c(B)+c'(A)}{c+c'}=\tfrac{1}{2}\pi cc' \text{ or } 0,$$

a result resembling Holditch's theorem in form.

On making (P) constant in (2), the locus of P is found to be a circle; thus Steiner's well-known theorem follows from (2) in the same way that the theorem given by Mr. Kempe at p. 190 follows from (1).

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ARITHMETICAL NOTE.

By Prof. *H. W. Lloyd Tanner, M.A.*

This note relates to numbers which have multiples of the form 100...001. If r be the radix of the scale of notation, N a given number, we seek the values of m and r which satisfy the congruence

$$r^m + 1 \equiv 0 \text{ (mod. } N) \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Clearly (1) is impossible unless N be prime to r.
Assuming this to be the case, we have

$$r^{\phi N} - 1 \equiv 0 \text{ (mod. } N) \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where ϕN is the number of numbers less than N and prime to N. Put

$$\phi N = 2^p M,$$

where M is odd. Then (2) may be written

$$(r^M - 1)(r^M + 1)(r^{2M} + 1)(r^{2^2 M} + 1) \ldots (r^{2^{p-1} M} + 1) \equiv 0 \text{ (mod. } N)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The factors on the left of (3) are prime to each other; save for a factor 2 common to all when r is odd.

I. Suppose, in the first place, that N is an odd prime or a power of an odd prime. Then only one of the elementary congruences implied in (3) can be satisfied; and, since (3) is true, one of these elementary congruences must be satisfied.

If $$r^M - 1 \equiv 0 \text{ (mod. } N) \ldots\ldots\ldots\ldots (4),$$

then (1) is impossible. For writing (1) in the form

$$r^m \equiv -1 \text{ (mod. } N)$$

and raising it to the odd power M, we have

$$r^{mM} \equiv -1 \text{ (mod. } N)$$

which is inconsistent with (4).

If (4) be not satisfied then some one of the other congruences implied in (3) must hold, say

$$r^{2^q M} + 1 \equiv 0 \text{ (mod. } N) \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

and this gives a multiple of the required form. It is to be noted that $2^q M$ is not necessarily the smallest value of m which satisfies (1). It is, however, easily proved that if (1) be true as well as (5), m must be an odd multiple of 2^q, and the smallest value of m is the product of 2_s into some factor of M. To prove the first of these statements observe that if

$$r^{2^q M} \equiv -1;$$

then, since M is odd,

$$r^{2^q M M_1} \equiv -1.$$

But raising (5) to the power M_1 we get

$$r^{2^q M M_1} \equiv \mp 1, \text{ according as } M_1 \text{ is } \begin{matrix}\text{odd}\\ \text{even}\end{matrix},$$

which is consistent with the former result only if M_1 is odd. In the same way if q' be less than q and M_1 be odd, the congruence

$$r^{2^{q'} M_1} + 1 \equiv 0$$

makes (5) impossible.

The second statement is proved when it is shown that two congruences

$$r^{2^q M_1} + 1 \equiv r^{2^q M_2} + 1 \equiv 0$$

imply a third

$$r^{2^q M} + 1 \equiv 0$$

where M is the G.C.M. of M_1, M_2. Now from the first pair we get

$$r^{2^q (hM_1 - kM_2)} \equiv \pm 1.$$

By properly choosing h, k, we can make

$$hM_1 - kM_2 = M.$$

This done, the lower sign only is consistent with either of the first two congruences.

II. Next take N to be a power of 2. Here r must be odd. Say it is $2^q r' - 1$ where r' is odd. Then N must be either 2 or $2^{q'}$ (where q' is not greater than q) according as m is even or odd.

For if m be even (1) becomes

$$(2^q r' - 1)^m + 1 = k 2^q + 2,$$

and this is divisible by no power of 2 but the first.

If m be odd,

$$(2^q r' - 1)^m - 1 = k2^{2q} + mr'2^q - 1 + 1,$$

and since m, r' are both odd, this is divisible by any power of 2 not greater than 2_q.

III. Lastly, if N be a composite number, say

$$N = 2^s . \alpha . \beta \ldots$$

where α, β,... are powers of odd primes.

Of course N must be prime to $r^M - 1$, save as to the factor 2^s, for if

$$r^M - 1 \equiv 0 \pmod{\alpha} \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

then α has no multiple of the form $r^m + 1$, and therefore N cannot have one.

When (6) is not true, then for some value of q we must have

$$r^{2^q M} + 1 \equiv 0 \pmod{\alpha}.$$

In this case if $r^m + 1$ be a multiple of α, m must be an odd multiple of 2^q. Therefore if $r^m + 1$ be a multiple of N, m must be an odd multiple of 2^q. Hence it follows that q must be the same for all the factors α, β....

As regards the even factors of N. If r be even, s must vanish. If r be odd s may be 0 or 1, unless q vanishes, when N must not be even in a higher degree than $r + 1$.

When N is a power of an odd prime, and M is small, it is easy to see whether N has or has not a multiple of the required form, by trying whether N divides $r^M - 1$ or not. For example, take the prime numbers of the form $2^p + 1$. Here $\phi N = 2^p$, $M = 1$; so that unless $2^p + 1$ happens to be a factor of $r - 1$, it has a multiple of the required form.

But this test gives no information as to the value of q, and is therefore useless in the case of composite numbers. To supply this deficiency observe that if (5) be satisfied, then r is a residue, with respect to N, of numbers raised to the power 2^{p-q-1}, but not to the power 2^{p-q}. For suppose r is a residue of the powers $2^{p'}$, or

$$r \equiv a^{2^{p'}} \pmod{N};$$

therefore

$$r^{2^{p-p'}} \equiv a^{2^p} \pmod{N};$$

therefore

$$r^{2^{p-p'}M} \equiv a^{2^p M} \equiv a^{\phi N} \equiv 1 \pmod{N}.$$

Comparing with (5), we see that r is a residue of the powers

$2^{p'}$ if $p-p'=q+1$ (or $p'=p-q-1$), but not when $p-p'=q$ (or $p'=p-q$).

These residues can at once be written down if only the quadratic residues of N are known. For if a_2 is the quadratic residence of a_1, and a_3 of a_2, then a_3 is the biquadratic residue of a_1; since if

$$a_1^2 \equiv a_2, \text{ and } a_2^2 \equiv a_3,$$

then

$$a_1^4 \equiv a_3.$$

In the examples which follow, the numbers are written in the denary scale unless otherwise stated.

Example 1. $N=7$, $\phi N=6$, $M=3$, $p=1$,

$q=p-1=0$. 1, 6 : 2, 5 : 3, 4,

1 : 4 : 2 .

Here 3, 5, 6 are non-residues of the squares, but of course residues of the first powers. For these numbers then q is $p-1$ or 0. In each of these scales 7 has a multiple r^3+1 or 1001. In the scale of 6 there is also a multiple $r+1$ or 11.

Example 2. $N=13$, $\phi N=12$, $M=3$, $p=2$,

$q=p-1=1$. 1, 12; 2, 11; 3, 10; 4, 9; 5, 8; 6, 7,

$q=p-2=0$. 1 ; 4 ; 9 ; 3 ; 12 ; 10 ,

1 ; 3 ; 3 ; 9 ; 1 ; 9 .

For 2, 5, 6, 7, 8, 11, $q=1$, and in each of these scales there is a multiple r^6+1 or 1000001 of N. In the scales 5, 8 there is a smaller multiple r^3+1 or 101.

For 4, 10, 12, $q=0$, so that r^3+1 or 1001 is a multiple of N in these scales. In the scale of 12, $r+1$ is also a multiple.

In the scales 1, 3, 9 there is no multiple of the required form.

Example 3. $N=17$, $\phi N=16$, $M=1$, $p=4$.

$q=p-1=3$. 1, 16 : 2, 15 : 3, 14 : 4, 13 : 5, 12 : 6, 11 : 7, 10 : 8, 9

$q=p-2=2$. 1 : 4 : 9 : 16 : 8 : 2 : 15 : 13

$q=p-3=1$. 1 : 16 : 13 : 1 : 13 : 4 : 4 : 16

$q=p-4=0$. 1 : 1 : 16 : 1 : 16 : 16 : 16 : 1

1 : 1 : 1 : 1 : 1 : 1 : 1 : 1

Here there is a multiple in every scale less than N but that of 1. In scale of 16, where $q=0$, there is a multiple 11. In

the scales of 4 and 13, 101 is a multiple. In the scales of 2, 8, 9, 15, for which $q=2$, there is a multiple 10001. And in the remaining scales (less than 17) r^8+1 is a multiple. Moreover, since M is 1, these are in each case the smallest multiples of the required form.

Example 4. $N=7.13$, $\phi N=72$, $M=9$, $p=3$.

Reference to Ex. 1 shews that for 7, q has only one value 0. The scales corresponding to this value are

$$3, 5, 6, 10, 12, 13 \ldots 7k+3, 7k+5, 7k+6, \ldots .$$

For $N=13$, Ex. 2 shews that for $q=0$ the values of r are

$$4, 10, 12, 17, \ldots 13k+4, 13k+10, 13k+12, \ldots .$$

Hence, N will have a multiple r^M+1 or 1000000001 in the scales of 10, 12, 17, &c. However, if we observe that for 7 and 13 M is 3, it is clear that we have in all of these scales a smaller multiple r^8+1 or 1001.

In a similar manner it may be found that $N=7.13.17$ has a multiple of the required form in the scale of 101; the lowest scale in which there is one.

April, 1878.

ELEMENTARY DEMONSTRATION OF TAYLOR'S THEOREM FOR FUNCTIONS OF AN IMAGINARY VARIABLE.

By Professor *Paul Mansion*.

I. *Fundamental formula.* Let $Fz=\phi(x, y)+i\psi(x, y)$ be a function of an imaginary variable $z=x+yi$, continuous, as also its derived function, from $z_0=x_0+y_0i$ to $Z=X+Yi$, when we pass from z_0 to Z by values of $x+yi$ such that x, y are the coordinates of a point of a curve $y=\chi x$, continuous also between the points $a(x_0, y_0)$ and $b(X, Y)$. We know that

$$\frac{\delta\phi}{\delta x}=\frac{\delta\psi}{\delta y}, \quad \frac{\delta\phi}{\delta y}=-\frac{\delta\psi}{\delta x}, \quad F'z=\frac{\delta\phi}{\delta x}+i\frac{\delta\psi}{\delta x}.$$

I say that

$$FZ-Fz_0=\int_{z_0}^{Z} F'z\,dz \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

the second member denoting the limit of the sum of all the expressions $F'z\,dz$, when x, y take successively the values corresponding to the different points of the curve ab, or $y=\chi x$, so that $dy=\chi' dx$. We have, in fact,

$$F'z\,dz=\left(\frac{\delta\phi}{\delta x}+i\frac{\delta\psi}{\delta x}\right)(dx+idy)$$

$$=\left(\frac{\delta\phi}{\delta x}dx-\frac{\delta\psi}{\delta x}dy\right)+i\left(\frac{\delta\psi}{\delta x}dx+\frac{\delta\phi}{\delta x}dy\right)$$

$$=\left(\frac{\delta\phi}{\delta x}+\frac{\delta\phi}{\delta y}\chi'\right)dx+i\left(\frac{\delta\psi}{\delta x}+\frac{\delta\psi}{\delta y}\chi'\right)dx$$

$$=\left(\frac{d\phi}{dx}+i\frac{d\psi}{dx}\right)dx,$$

whence $$\int_{z_0}^{Z}F'z\,dz=\int_{x_0}^{X}\left(\frac{d\phi}{dx}+i\frac{d\psi}{dx}\right)dx$$

$$=\Big[\phi(x,y)+i\psi(x,y)\Big]_{x_0,y_0}^{X,Y}=FZ-Fz_0.$$

II. *Taylor's theorem.* Putting in this formula

$$Fz=fZ-\left[fz+\frac{Z-z}{1}f'z+\frac{(Z-z)^2}{1.2}f''z+\ldots+\frac{(Z-z)^{n-1}}{1.2\ldots(n-1)}f^{n-1}z\right],$$

we have Taylor's, or rather Cauchy's, theorem,

$$fZ=fz_0+\frac{Z-z_0}{1}f'z_0+\ldots+\frac{(Z-z_0)^{n-1}}{1.2\ldots(n-1)}f^{n-1}z_0+r_n,$$

$$r_n=\int_{z_0}^{Z}\frac{(Z-z)^{n-1}}{1.2\ldots(n-1)}f^n z\,dz.$$

III. *Forms of the remainder given by MM. Falk and Darboux.* Denote by p, p_1, p_2, three positive arbitrary quantities not superior to n, by θ, θ_1, θ_2 quantities comprised within 0 and 1, by μ a quantity not negative and not superior to unity, by ν an indeterminate angle.

Write also, with Weierstrass, $R(\alpha+\beta i)$ for the real part α of an imaginary expression $\alpha+\beta i$, $I(\alpha+\beta i)$ for the imaginary part βi. Taking for the curve $y=\chi x$ the straight line

which joins the points a and b, we can put the remainder r_n in either of the following forms:

$$r_n = \mu e^{\nu i} \frac{(Z-z_0)^n}{1.2\ldots(n-1)} \frac{(1-\theta)^{n-p}}{p} f^n[z_0+\theta(Z-z_0)],$$

$$r_n = R\left[\frac{(Z-z_0)^n}{1.2\ldots(n-1)} \frac{(1-\theta_1)^{n-p_1}}{p_1} f^n z_1\right] + I\left[\frac{(Z-z_0)^n}{1.2\ldots(n-1)} \frac{(1-\theta_2)^{n-p_2}}{p_2} f^n z_2\right],$$

$$z_1 = z_0+\theta_1(Z-z_0), \quad z_2 = z_0+\theta_2(Z-z_0).$$

The first is due to M. Darboux (Liouville's Journal, continued by Resal, September, 1876), who obtains it by a different method. The second is due to M. Falk (memoir presented to the Society of Sciences of Upsala, July 19, 1877); it leads more easily, than does M. Darboux's form, to the fundamental formula relative to the true values of expressions that assume the form $\frac{0}{0}$.

IV. *Applications.* By means of the last forms of the remainder we readily establish, in an elementary manner, in the cases where they subsist, the developments in infinite series of the functions e^z, $\log(1+z)$, $(1+z)^m$, z being of the form $x+yi$.

V. *Demonstration of M. Falk's formula, by Homersham Cox's method.* Having recourse only to Rolle's theorem we can demonstrate without difficulty the formula

$$FZ - Fz_0 = R[(Z-z_0)F'z_1] + I[(Z-z_0)F'z_2] \ldots (2),$$

analogous to (1), but less precise. Applying this relation to the function

$$Fz = fZ - \left[fz + \frac{Z-z}{1} f'z + \frac{(Z-z)^2}{1.2} f''z + \ldots + \frac{(Z-z)^{n-1}}{1.2\ldots(n-1)} f^{n-1}z\right] - (Z-z)^p P,$$

where P is such that $Fz_0 = 0$, and p successively equal to p_1 and to p_2, we obtain again Taylor's theorem, with M. Falk's form of the remainder.

This demonstration does not essentially differ from that of the Swedish mathematician, for he obtains Taylor's formula

by adding the development of the real part of the function to that of the imaginary part. And the formula (2) has been obtained by the same procedure.

VI. *Conclusion.* It follows from what precedes that we can deduce the complete theory of the elementary functions (functions expressible algebraically by radicals, singly periodic transcendental functions and their inverses), *the variable being real or imaginary*, from Rolle's theorem or from the geometrical lemma equivalent to it, "If a plane curve AIB, continuous between the points AB and whose tangent changes continuously between their two points, is cut by the secant AB, there is an intermediate point I where the tangent is parallel to AB."

Antwerp, *May*, 1878.

EXAMPLE ILLUSTRATIVE OF A POINT IN THE SOLUTION OF DIFFERENTIAL EQUATIONS IN SERIES.

By *J. W. L. Glaisher.*

In a paper "Note on the integration of certain differential equations by series,"* Prof. Cayley has directed attention to a curious point in the solution of differential equations in series, viz. that if we have a solution

$$A\left(x^{\alpha}+\frac{a_1}{b_1}x^{\alpha+1}+\frac{a_1a_2}{b_1b_1}x^{\alpha+2}+\&c.\right),$$

then if one of the factors in a numerator, say a_r, vanishes, we may stop at the preceding term, the finite series

$$A\left(x^{\alpha}+\frac{a_1}{b_1}x^{\alpha+1}\ldots+\frac{a_1a_2\ldots a_{r-1}}{b_1b_2\ldots b_{r-1}}x^{\alpha+r}\right),$$

being a particular integral; but that if we continue the series notwithstanding this evanescent factor, and if a factor in a denominator, say b_s, $(s>r)$ vanishes, then the series recommences with the term involving $x^{\alpha+s}$, and we have another particular integral

$$A'\frac{0}{0}\left(x^{\alpha+s}+\frac{a_{s+1}}{b_{s+1}}x^{\alpha+s+1}+\&c.\right),$$

in which $A'\frac{0}{0}$ may be replaced by a new arbitrary constant B. Thus if the law of formation of the coefficients of the

* *Messenger of Mathematics*, Old Series, vol. v., pp. 77—83 (1869).

terms is such that a factor in a numerator first vanishes and subsequently a factor in a denominator, we have a particular integral consisting of a finite number of terms, and the series recommences and gives another particular integral.

Prof. Cayley illustrates this in his paper, and in what follows I merely propose to give another example of a differential equation whose solution in series involves the point in question in a simple form.

Consider the differential equation

$$x(1-4x)\frac{d^2u}{dx^2}+\{(4p-6)x-p+1\}\frac{du}{dx}-p(p-1)u=0.$$

To integrate this in series, put

$$u=\Sigma A_r x^{m+r},$$

and by substitution in the differential equation we find that the relation between A_r and A_{r+1}, is

$$(m+r+1)(m+r-p+1)A_{r+1}-(2m+2r-p)(2m+2r-p+1)A_r=0.$$

Let A_{-1}, A_{-2}... be each equal to zero, and putting $r=-1$, the values of m are given by the equation

$$m(m-p)A_0=0,$$

viz. $m=0$, or $m=p$. Taking the root $m=0$, we have

$$A_1=-\frac{p.p-1}{1.p-1}A_0,$$

$$A_2=-\frac{p-2.p-3}{2.p-2}A_1,$$

$$A_3=-\frac{p-4.p-5}{3.p-3}A_2, \text{ &c.,}$$

whence the particular integral corresponding to the root $m=0$ is

$$P=1-\frac{p.p-1}{p-1}x+\frac{p.p-1.p-2.p-3}{p-1.p-2}\frac{x^2}{2!}+\frac{p.p-1\ldots p-5}{p-1.p-2.p-3}\frac{x^3}{3!}-\text{&c.};$$

and, similarly, the particular integral corresponding to $m=p$ is

$$Q=x^p\left\{1+\frac{p.p+1}{p+1}x+\frac{p.p+1.p+2.p+3}{p+1.p+2}\frac{x^2}{2!}+\text{&c.}\right\}.$$

Now suppose p not $=\pm$ an integer; these series are independent, neither terminates, and the general integral

of the differential equation is $AP+BQ$, A and B being arbitrary constants.

Suppose $p=$ a positive integer, n, then P consists of a finite portion

$$P'=1-\frac{n\,.\,n-1}{n-1}x+\frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{n-1\,.\,n-2}\,\frac{x^2}{2\,!}-\cdots,$$

the last term being

$$(-)^{\frac{1}{2}n}\frac{n\,!}{n-1\,.\,n-2\ldots\frac{1}{2}n}\,\frac{x^{\frac{1}{2}n}}{(\frac{1}{2}n)\,!},\ =(-)^{\frac{1}{2}n}\,2x^{\frac{1}{2}n},$$

if n be even, and $(-)^{\frac{1}{2}(n-1)}\,nx^{\frac{1}{2}(n-1)}$ if n be uneven.

There are then $\frac{1}{2}n-1$ or $\frac{1}{2}(n-1)$ zero terms according as n is even or uneven, and the series recommences with the term in x^n, and is

$$(-)^n\frac{n\,!\,0\,.-1\,.-2\ldots-(n-1)}{n-1\,.\,n-2\ldots1\,.\,0}\,\frac{x^n}{n\,!}\left\{1+\frac{n\,.\,n+1}{n+1}x+\&c.\right\}$$

$$=-Q,$$

if we divide out the factor 0 or $n-n$ from the numerator and denominator. Thus, when the series recommences, it gives the other particular integral, and the general integral may be obtained from P alone and is $AP'+BQ$.

Similarly if $p=$ a negative integer, $Q=Q'-P$, where Q' denotes the finite part of the series continued up to the term immediately preceding the first term that contains a zero factor in a numerator, and the general integral is $AQ'+BP$.

If p be unrestricted we have the two particular integrals P and Q, but if p be a positive integer $P=P'-Q$* so that we may take the two particular integrals to be P' and Q, and the point to be specially noticed is this: Knowing that, for all values of p, P, extending to infinity, is a particular integral, it might seem that when P stops and recommences, it would be necessary to include the whole series up to infinity, so that the particular integral would be $P'+R$, and we should have, not a finite integral, but an infinite series with certain terms omitted; but in reality we *may* take the finite portion P' as a new particular integral, and R must be of the form, const. $\times\,Q$.

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* The algebraical identity $P=P'-Q$ is readily proved by putting $p=n+h$, dividing out the factor h when it occurs in both numerator and denominator, and making h indefinitely small. It is this consideration that justifies the dividing out by 0 or $n-n$ above.

In this case the two infinite series P, Q admit of summation, for

$$P = 1 - px + \frac{p \cdot p-3}{2\,!}x^2 - \frac{p \cdot p-4 \cdot p-5}{3\,!}x^3 + \&c.$$
$$= \{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{(1-4x)}\}^p,$$
$$Q = \{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{(1-4x)}\}^p;$$

and, as is well known,* when p = a positive integer,

$$P + Q = 1 - px + \frac{p \cdot p-3}{2\,!}x^2 - \&c.,$$

the series extending to $\frac{1}{2}p+1$ or $\frac{1}{2}(p+1)$ terms according as p is even or uneven, and if p = a negative integer,

$$P + Q = x^p\left\{1 + px + \frac{p \cdot p+3}{2\,!}x^2 + \&c.\right\},$$

the series extending to $-\frac{1}{2}p+1$ or $\frac{1}{2}(-p+1)$ terms according as p is even or uneven.

If P and Q had not admitted of summation, we should have had the more usual case of the differential equation not possessing a finite particular integral except in the cases when one of the series terminates through the presence of a zero factor in a numerator. This happens in the solution of Riccati's equation, $\frac{d^2u}{dx^2} - x^{m-2}u = 0$, and its numerous transformations, where the series terminate when $q = \pm$ the reciprocal of an uneven integer, and we have two finite particular integrals in each case, so that the equation is finitely integrable for these forms of q, and for these forms only, as is well known. But the above example illustrates equally well, and in a simpler form, the points in the theory already alluded to, and also how it happens that an equation may admit of finite integration in certain cases and not in others.

NOTE ON THOMSON'S THEORY OF THE TIDES OF AN ELASTIC SPHERE.

By *G. H. Darwin, M.A.*

THE results of the theory of the elastic yielding of the earth would of course be more interesting, if it were possible to fully introduce the effects of the want of homogeneity of elasticity and density of the interior of the earth; but besides

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* Todhunter's *Differential Calculus*, chapter IX.

the mathematical difficulties of the case, the complete absence of data as to the nature of the deep-seated matter makes it impossible to do so. It is, however, possible to make a more or less probable estimate of the extent to which a given yielding of the *surface* will affect the ocean tide-wave, when the earth is treated as heterogeneous. And as we can only judge of the amount of the bodily tide in the earth by observations on the ocean tides, this estimate may be of some value.

The heterogeneity of the interior must of course be accompanied by heterogeneity of elasticity,* and under the influence of a given tide generating force, this will affect the internal distribution of strain, and the form of the surface to an unknown extent. The diminution of ocean tide which arises from the yielding of the nucleus is entirely due to the alteration in the form of the level surfaces outside the nucleus. But it is by no means obvious how far the potential of the earth, when its surface is distorted to a given amount, may differ from that of the homogeneous spheroid considered by Sir W. Thomson; and in face of our ignorance of the law of internal elasticity, the problem does not admit of a precise solution.

I propose, however, to make an hypothesis, which seems as probable as any other, as to the law of the ellipticity of the internal strain ellipsoids, when the surface is strained to a given amount, and then to find the potential at an external point.

Suppose, then, that under the influence of a bodily harmonic potential of the second degree the earth's surface assumes the form $r = a + \sigma$, where σ is a surface harmonic of the second order. Then I propose to assume that the ellipticity of any internal strain ellipsoid is related to that of the surface by the same law as though the earth were homogeneous, elastic, and incompressible, and had its surface brought into the form $r = a + \sigma$ by a tide generating potential of the second order. If μ be the coefficient of rigidity of an elastic incompressible sphere under the action of a bodily force, of which the potential is wr^2S_2, then Sir W. Thomson's solution† shews that the radial displacement at any point r is given by

$$\rho = \frac{8a^2 - 3r^2}{19\mu} rS_2.$$

* That is to say, if the earth is elastic at all.

† Thomson and Tait's *Natural Philosophy*, § 834, equation. (14).

Putting $r=a$, we have $\sigma=\frac{5a^3}{19\mu}S_2$. And if $r=a'+\sigma'$ be the equation to a strain ellipsoid of mean radius a', we have by our hypothesis

$$\frac{\sigma'}{a'}\div\frac{\sigma}{a}=\frac{8a^2-3a'^2}{5a^2},$$

and $$\sigma'=\frac{a'}{a}\left\{\tfrac{8}{5}-\tfrac{3}{5}\left(\frac{a'}{a}\right)^2\right\}\sigma=f(a')\,\sigma \text{ suppose.}$$

Now the potential of a homogeneous spheroid $r=a'+\sigma f(a')$, of density q, at an external point is

$$\tfrac{4}{3}\pi q\frac{a'^3}{r}+\frac{4\pi q}{5r^3}a'^3f(a')\,\sigma,$$

and therefore the potential of a spheroidal shell of density q, whose inner and outer surfaces are given by $r=a'+f(a')\,\sigma$ and $r=a'+\delta a'+f(a'+\delta a')\,\sigma$, is

$$4\pi q\frac{a'^2}{r}\delta a'+\frac{4\pi q}{5r^3}\frac{d}{da'}\{a'^3f(a')\}\,\sigma\delta a'.$$

If then we integrate this expression from $a'=a$ to $a'=0$, and treat q as a function of a', we have the potential of the earth on the present hypothesis. The integral is

$$\frac{4\pi}{r}\int_0^a qa'^2da'+\frac{4\pi\sigma}{5r^3}\int_0^a q\frac{d}{da'}\{a'^3f(a')\}\,da'.$$

The first of these two terms is clearly $g\frac{a^2}{r}$ (where g is gravity), and is the same as though the earth were homogeneous; and it only remains to evaluate the second. Now, according to the Laplacean law of internal density of the earth, if D be the mean density, and f the ratio of D to the surface density, and θ a certain angle which is about 144°,

$$q=\frac{D}{f}\frac{a}{\sin\theta}\cdot\frac{\sin\frac{a'\theta}{a}}{a'}.$$

Substituting this value for q, and for $f(a')$ its value, we have

$$q\frac{d}{da'}[a'^3f(a')]\,da'=\tfrac{2}{5}\frac{D}{f\sin\theta}\frac{\sin\frac{a'\theta}{a}}{a'}\left(16a'^3-\frac{9a'^5}{a^2}\right)da'.$$

Then if we change the variable of integration by putting

$x=\frac{a'\theta}{a}$, and put $4\pi D=\frac{3g}{a}$, we get for the second term of the earth's potential

$$\left\{\tfrac{3}{5}g\left(\frac{a}{r}\right)^2\sigma\right\}\frac{2}{5f\theta^3\sin\theta}\int_0^\theta \sin x\left(16x^2-\frac{9x^4}{\theta^2}\right)dx.$$

Now $\frac{3}{5}g\left(\frac{a}{r}\right)^2\sigma$ would be the potential of the surface layer given by $r=a+\sigma$, *if the earth were homogeneous and had a density* D, and the rest of the expression is a numerical factor (which may be called K), by which this potential must be reduced in order to get the potential of the heterogeneous earth on the present hypothesis.

If the integration be effected it will be found that

$$\tfrac{1}{2}fK=-\cot\theta\left[\frac{7}{\theta}+\frac{76}{\theta^3}-\frac{216}{\theta^5}\right]-\operatorname{cosec}\theta\left[\frac{32}{\theta^3}+\frac{216}{\theta^5}\right]-\frac{4}{\theta^2}+\frac{216}{\theta^4}$$
$$=5{\cdot}1442, \text{ when } \theta=144^\circ.$$

Whence $$K=\frac{2{\cdot}0577}{f}.$$

Also $$f=3\left(\frac{1}{\theta^2}-\frac{\cot\theta}{\theta}\right)=2{\cdot}1178 \text{ by Laplace's theory.}$$

Therefore $$K=\frac{2{\cdot}0577}{2{\cdot}1178}=\cdot 972.$$

Hence, on the present hypothesis, the potential of the earth at a point outside its mass is

$$g\frac{a^2}{r}+(\cdot 972)\,\tfrac{3}{5}g\left(\frac{a}{r}\right)^2\sigma.$$

This differs by very little from what it would be if the earth were homogeneous; for in that case ·972 would be merely replaced by unity.

Therefore, if at any future time it should be found that the fortnightly tide* is less than it would be theoretically on a rigid nucleus, it will then be probable that the surface of the earth rises and falls by about the same amount as would follow from the theory of the bodily tides of a *homogeneous* elastic sphere whose density is equal to the earth's mean density. This investigation being founded on conjecture, cannot claim anything better than a probability for its result; but without calculation, I, at least, could not form any sort of guess of what the result might be, and the question is of undoubted interest in the physics of the earth.

* Sir W. Thomson relies principally on observation of the fortnightly ocean tide for detecting bodily tides in the earth.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, April 11th, 1878.—C. W. Merrifield, Esq., F.R.S., *Vice-President*, in the Chair. Mr. Artemas Martin, M.A., Erie, Pennsylvania, was elected a member; Mr. W. M. Hicks, M.A., and Mr. T. R. Terry, M.A., were proposed for election. The Chairman, on the recommendation of the Council, nominated Professors Brioschi (Milan), Darboux (Paris), Gordan (Erlangen), Sophus Lie (Christiania), and Mannheim (Paris), as Honorary Foreign Members of the Society, the Council having decided to increase the number of foreign members from seven to twelve. Prof. H. J. S. Smith, F.R.S., read two papers, "Second notice on the characteristics of the Modular Curves" and a "Note relating to the Theory of the Division of the Circle." Prof. Cayley spoke on the subject of both papers, asking, in the course of his remarks, if a solution had been effected for the inscription of a regular heptagon, assuming the trisection of an angle. Mr. Tucker read a letter from Prof. Tait, in which he stated that by a simple *physical* process he could easily obtain any number of definite integrals similar to the following:

$$\int_0^{\pi}\int_0^{\pi} \sin\theta\, d\phi\, d\theta \, \frac{r^2 - a^2 - \lambda^2 + 2a\lambda\cos\theta}{[r^2+a^2+\lambda^2-2a\lambda\cos\theta-2r\lambda\cos\alpha+2ar(\cos\alpha\cos\theta-\sin\alpha\sin\theta\cos\phi)]^{\frac{3}{2}}}$$
$$= \frac{2\pi(r^2-\lambda^2)}{(r^2+\lambda^2-2r\lambda\cos\alpha)^{\frac{1}{2}}}.$$

Prof. Tait said he wished to know whether the solution could be easily effected by direct analytical processes. Mr. Tucker then read an abstract of a paper by Prof. Minchin, "On the astatic conditions of a body acted on by given forces." When a body is acted on by given forces applied at given points in the body, if it is in equilibrium, it will under certain conditions remain so when it is displaced in any manner, each force retaining its magnitude, direction, and point of application in the body. The requisite conditions are called the astatic conditions. The investigation of them by ordinary Cartesian methods is given in Moigno's *Statique* at great length. Prof. Minchin's paper treats them by elementary quaternion methods, and adds a few geometrical results not noticed by Moigno. The paper also contains a proof of Minding's theorem, viz. that in certain positions of displacement of the body, the given forces reduce to a single resultant force; and when this happens, the line of action of the single force intersects both focal conics of a certain quadric, having for centre the centre of the central plane, and this plane for one of its principal planes. He then read part of a paper by Mr. C. Leudesdorf, "On certain extensions of Frullani's Theorem." The object of the paper is to supplement two papers communicated by Mr. E. B. Elliott, and printed in the Society's *Proceedings*.

Thursday, May 9th.—Lord Rayleigh, *President*, in the Chair. Messrs. W. M. Hicks and T. R. Terry were elected Members, and Prof. G. M. Minchin was admitted into the Society. Professors Brioschi, Darboux, Gordan, Sophus Lie, and Mannheim were elected Honorary Foreign Members. Prof. Henrici, F.R.S., communicated a paper by Dr. Klein, of Munich, "Ueber die Transformation der Elliptischen Functionen;" Prof. Cayley, F.R.S., gave an account of a paper on the "Theory of Groups;" Prof. Kennedy read a paper "On some new solutions for certain problems in the Statics of Mechanisms." The special object of the paper was to give an elementary solution of the problem: Given a linkwork or plane mechanism of any number of links, with any force acting on any one of them, to find the magnitude of the force necessary to balance the mechanisms if acting in any direction on any other link. The method employed was the replacement of the two links on which the forces acted by two others, which had the same instantaneous centres and the same angular-velocity-ratio, but which were so chosen that they could be directly connected together by a third link. In this way a simple combination of three links was used as a "critical mechanism" to replace the original complex linkwork, and the solution became extremely simple. Incidentally the author took occasion to insist on the advantages of the consistent use of the action of the instantaneous centre even in the most elementary treatments of mechanical problems. Mr. J. W. L. Glaisher, F.R.S., briefly gave the principal results contained in his paper "Generalized Form of Certain Series," and Mr. Kempe spoke "On Conjugate Four-piece Linkages."

R. TUCKER, M.A., *Hon. Sec.*

AN ENUMERATION OF PRIME-PAIRS.

By *J. W. L. Glaisher.*

§ 1. THE present paper contains the tables of the number of prime-pairs alluded to in the last paragraph of my paper "On Long Successions of Composite Numbers" (vol. VII., p. 176). I call a pair of primes separated by only one number a *prime-pair*; thus 11 and 13, 17 and 19, 29 and 31, &c., are prime-pairs. It is clear that, as the number of primes decreases, as we ascend higher in the series of numerals, the number of prime-pairs must decrease also, and I was desirous of examining the rapidity of this decrease. I, accordingly, had the number of prime-pairs in each thousand or chiliad, in each of the first hundred thousand numbers in each of the six millions over which Burckhardt's and Dase's tables extend, counted: the results are arranged in the following tables.

The explanation of the tables is as follows: The number of prime-pairs between 0 and 1,000 is 36, between 1,000 and 2,000 is 26, between 2,000 and 3,000 is 21, ... between 10,000 and 11,000 is 16, and so on. Thus, for example, the number of prime-pairs between 83,000 and 84,000 is 8, between 1,064,000 and 1,065,000 is 7, between 8,041,000 and 8,042,000 is 3, &c. When a prime-pair extends over two chiliads, viz. when it is ...999 and ...001, it is counted as belonging to the lower of the two chiliads, and the number of prime-pairs in that chiliad is marked with an asterisk. Thus the prime-pairs between 1,028,000 and 1,029,000 are

$$\left.\begin{matrix}1{,}028{,}099\\101\end{matrix}\right\}, \left.\begin{matrix}189\\191\end{matrix}\right\}, \left.\begin{matrix}327\\329\end{matrix}\right\}, \left.\begin{matrix}471\\473\end{matrix}\right\}, \left.\begin{matrix}579\\581\end{matrix}\right\}, \left.\begin{matrix}681\\683\end{matrix}\right\}, \left.\begin{matrix}747\\749\end{matrix}\right\},$$

$$\left.\begin{matrix}939\\941\end{matrix}\right\}, \left.\begin{matrix}1{,}028{,}999\\1{,}029{,}001\end{matrix}\right\},$$

and are 9 in number, the 9 being marked by an asterisk; and the last prime-pair is not counted in the 4 prime-pairs between 1,029,000 and 1,030,000, which are

$$\left.\begin{matrix}1{,}029{,}359\\361\end{matrix}\right\}, \left.\begin{matrix}407\\409\end{matrix}\right\}, \left.\begin{matrix}839\\841\end{matrix}\right\}, \left.\begin{matrix}881\\883\end{matrix}\right\}.$$

It should be stated that in the first chiliad of the first million 1 and 3, 3 and 5, 5 and 7 are counted as prime-pairs, the number 2 being ignored.

The enumeration was made in duplicate from Burckhardt's and Dase's tables, and the two calculations were compared and brought into agreement, so that the tables ought to be free

from error. One enumeration of the portion from 0 to 100,000 was made from Burckhardt and the other from Chernac.

First Million.

	0	1	2	3	4	5	6	7	8	9
	36	26	21*	21	23	17	19	13	15*	15
1	16	14	11	15	11	12	13	18	12	15
2	15	15	16	14	6	12	11	15	12	9
3	11	13	19	13	16	11	9	12	11	9
4	8	12	12	11	11	7	12	11	13	17
5	9	14	8	11	8	11	14	9	11	11
6	10	6	8	10	9	14	6	10	9	12
7	16	10	11	6	10	9	11	9	9	11
8	13	12	13	8	12	11	8	14	10	8
9	10	10	12	12	13	8	7	15	13	8

Number of prime-pairs between 0 and 100,000 = 1,225.

Second Million.

	0	1	2	3	4	5	6	7	8	9
	11	6	9	9	4	9	11	7	10	8
1	7	5	8	9	6	6	10	8	4	6
2	8	10	5	9	4	7	10	7	9*	4
3	7	7	7	7	13	4	9	3	7	5
4	9	9	7	8	6	9	7	11	6	7
5	7	6	7	6	6	6	6	6	8*	4
6	6	4	9	8	7	6	5	5	11	8
7	6	8	5	9	11	9	5	6	6	6
8	9	9	6	8	2	4	9	6	8	4
9	12	12	8	10	8	8	6	7	5	8

Number of prime-pairs between 1,000,000 and 1,100,00 = 725.

Third Million.

	0	1	2	3	4	5	6	7	8	9
	6	6	7	6*	5	6	6	16	4	5
1	3	6	8	6	8	8	9	4	7	6
2	7	6	10	5	9	6	5	8	8	11
3	8	2	11	8	3	6	7	8	5	5
4	8	1	3	6	7	8	6	7	1	8
5	6	10	5	7	7	5	7	8	6	6
6	9	4	8	9	6	5	8	5	7	5
7	8	5	5	7	7	8*	3	5	5	11
8	4	4	5	7	7	6	11	7	12	6
9	5	5	8	3	5	3	8	7	6	3

Number of prime-pairs between 2,000,000 and 2,100,000 = 644.

Seventh Million.

	0	1	2	3	4	5	6	7	8	9
	6	2	6	7	10	8	5	8	5	7
1	8	6	6	6	5	7	6	6	4	8
2	5	5	6	6	8	4	6	6	4	4
3	6	4	2	6	4	8*	1	7	3*	4
4	5	9	7	7	6	5	2	6	5	4
5	5	10	4	4	5	5	6	4	8	6
6	4	2	5	5	1	4	9	4	4	7
7	3	7	7	7	4	7	7	3	7	3
8	8	5	6	6	7	3	4	8	8	5*
9	6	4	2	3	5	5	8	5	5*	4

Number of prime-pairs between 6,000,000 and 6,100,000 = 545.

Eighth Million.

	0	1	2	3	4	5	6	7	8	9
	3	4	6	7	5	5	3	4	7	6
1	6	2	7	3	6	3	6	6	9	5
2	6	5	9*	3	5	5	5	5	7	3
3	5	8	5	5	7*	6	7	3	6	5
4	7	6	4	5	7	8	7	9	6	4
5	2	6	5	3	4	3	7	5	5	6
6	3	6	4	2	5	5	4	3	6	3
7	4	9	5	5	4	4	3	8	4	8
8	3	9	5	3	4	3	3	6	6	9
9	6	6	5	6	6	5	8	9	4	2

Number of prime-pairs between 7,000,000 and 7,100,000 = 525.

Ninth Million.

	0	1	2	3	4	5	6	7	8	9
	13	4	4	6	7	3	9	6	4	11
1	4	3	6	6	0	6	4	7	4	5
2	4	7	2	4	4	2	9	7	4	4
3	4	8	5	6	1	4	1	6	4	4
4	6	3	4	4	7	8	6	6	8	6
5	4	2	3	6	7	8	7	5	4	8
6	5	7	5	4	5	6	5	7	6	5
7	9	6	5	1	5	7	5	5	11	7
8	3	4	8	4	3	6	2	5	5	5
9	4	1	4	5	4	6	4	2	5	8

Number of prime-pairs between 8,000,000 and 8,100,000 = 518.

§ 2. In the following table is shown the number of

chiliads, in the first hundred chiliads of each million, each of which contains no prime-pair, 1 prime-pair, 2 prime-pairs, &c.

Thus, for example, of the hundred chiliads between 6,000,000 and 6,100,000 two chiliads contain each 1 prime-pairs, twenty contains each 6 prime-pairs, while only two contain 10 prime-pairs.

Number of chiliads, each of which contains n prime-pairs, in the first hundred chiliads of each million.

n	0 to 100,000	1,000,000 to 1,100,000	2,000,000 to 2,100,000	6,000,000 to 6,100,000	7,000,000 to 7,100,000	8,000,000 to 8,100,000
0						1
1			2	2		4
2		1	1	5	4	5
3		1	7	6	17	6
4		9	5	19	13	26
5		8	19	19	23	17
6	4	21	20	20	21	17
7	2	16	16	14	10	11
8	9	15	18	11	5	7
9	10	16	4	2	7	3
10	8	5	2	2		
11	17	5	4			2
12	13	2	1			
13	9	1				1
14	6					
15	8					
16	4		1			
17	2					
18	1					
19	2					
21	2					
23	1					
26	1					
36	1					

The numbers of prime-pairs, as given by the above tables, are rather less than one-tenth of the numbers of primes in the same intervals. For the numbers of primes between 0 and 100,000, 1,000,000 and 1,100,000, ... 8,000,000 and 8,100,000, are respectively 9,593, 7,216, 6,874, 6,397, 6,369, 6,250. The number of primes between 0 and 10,000 (including both 1 and 2 as primes) is 1230, and it is a curious coincidence that this is so nearly equal to the number of prime-pairs between 0 and 100,000, viz. 1,225.

There can be little or no doubt that the number of prime-pairs is unlimited; but it would be interesting, though probably not easy, to prove this.

§ 3. On p. 106, vol VII., a proof is given that any sequence of composite numbers, however long, must occur at a definite place in the series of numerals, for if p, q be two consecutive primes, the $q-2$ numbers immediately following $(2.3.5.7...p)+1$ are composite. This is at once seen to be true, but the reason for it may be more fully explained as follows:

Consider the process of formation of a factor table, which shall give the least-factor of every number, but such least-factor not exceeding a given prime p. The entries in such a table will be periodic, the period being $2.3.5.7...p$. For, to fix the ideas, imagine the table to consist of an infinite strip, the natural numbers being written in order at equal distances from another, and the least-factor of each being placed opposite to it, as below:

2	3	2	5	2	7	2	3	2	11	2	13	2	3	2	17	2	19	2	...
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	...

To construct such a table we place a 2 opposite every second number, starting from the beginning; we may then place a 3 opposite every third number, but we must then erase the 3 from every sixth number which will have 2 as its least-factor. Thus, as far as the factors 2 and 3 are concerned, the table is periodic, the period being 6, or 2.3. Similarly, if we place a 5 opposite to every fifth number, we must erase it from every tenth and fifteenth number, and we see that, as far as the least-factors 2, 3, 5 are concerned, the table repeats itself after every 2.3.5 numbers. This reasoning is clearly general, and if we enter all the least-factors not exceeding p, the table repeats itself after $2.3.5.7...p$ numbers. But observe that the first time any least-factor appears in the table, it is equal to the number opposite to which it is written, that number being a prime, viz. the first time any least-factor appears in the table, it indicates a prime, and whenever it appears subsequently it indicates a composite number.

Now if we form a factor table, entering all least factors not exceeding p, then the numbers $2, 3, 4, 5, ...q-1$, where q is the next prime to p, will have a least-factor opposite to each of them, and the table repeats itself with a period of $(2.3.5.7...p)$,

or say, for brevity, P; it follows that the numbers $P+2$, $P+3$, $P+4$, ... $P+q-1$ will have a least factor opposite to each of them. The fact that the numbers 2, 3, 4, ...$q-1$ have a factor opposite to each of them implies nothing in regard to a sequence of composite numbers, as this will be the first appearance of many of the least-factors; but when they occur again opposite to $P+2$, $P+3$, ... $P+q-1$, each least factor indicates a composite number, and we have a sequence of $q-1$ composite numbers. As the number $P+1$ corresponds to 1, it may or may not be really prime, but if only least-factors not exceeding p be entered, it will have a blank opposite to it.

Practically, of course, in a factor table when the least-factor is equal to the number itself, the latter is prime and no least-factor is entered; but in considering such a table theoretically, it is convenient to suppose the least-factor always entered, in which case the primes correspond to first appearances of least factors, and the composite numbers to subsequent appearances.

A glance at the above specimen table shows that if we suppose only 2's and 3's entered as least-factors, then, as the table repeats itself with period 6, only places opposite numbers of the form $6n \pm 1$ will be blank, *i.e.* that, for numbers above 6, only numbers of this form can be primes. This is of course proved at once, and is well known, but it is noticed here as the most simple consequence of the *periodicity*, which is a fundamental property of factor-tables.

ON SOME THEOREMS IN DETERMINANTS.

By *R. F. Scott, M.A.*, St. John's College.

1. If when $x=a$ the elements of p columns of a determinant become identical, the determinant is divisible by $(x-a)^{p-1}$.

For subtract one of the columns from each of the remaining $p-1$ others, then when $x=a$ the elements of these columns vanish, and hence each is divisible by $x-a$. The value of the determinant not having been altered by this operation the truth of the theorem is at once manifest. The theorem is also clearly true when the elements of the p columns do not become identical but are only proportional.

Ex. If $a_1, a_2 \ldots a_n$ are any n quantities whose sum is S, and if $A_m = S - a_m$, then

$$D = \begin{vmatrix} x - A_1, & a_2 & , \ldots a_n \\ a_1 & , x - A_2, & \ldots a_n \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_1 & , a_2 & , \ldots x - A_n \end{vmatrix} = x(x - S)^{n-1}.$$

For, adding all the columns to the first, the elements of that column are all equal to x, thus x is a factor of the determinant. Putting $x = S$, all the elements of the i^{th} column are a_i, whence, by the above theorem, the determinant divides by $(x - S)^{n-1}$.

Thus $$D = kx(x - S)^{n-1},$$

where k is a factor, which we see is unity by comparing the coefficients of x^n.

This theorem was given by M. Lucas, *Comptes Rendus*, 1870, p. 1167, without proof, and has been proved less briefly by M. Albeggiani, *Giornale*, XIII. p. 107, and by M. Guenther, *Zeithschrift*, XXI. p. 178, and *Erlangen Sitzungsberichte*, 23 June, 1873.

In the same way it follows that

$$\begin{vmatrix} x - a_1, & A_2 & , \ldots A_n \\ A_1 & , x - a_2, & \ldots A_n \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ A_1 & , A_2 & , \ldots x - a_n \end{vmatrix} = \{x - (n-2)S\}(x - S)^{n-1}.$$

2. Determinants of the form

$$D = \begin{vmatrix} x_1, & x_2, & x_3, & \ldots x_n \\ x_n, & x_1, & x_2, & \ldots x_{n-1} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ x_2, & x_3, & x_4, & \ldots x_1 \end{vmatrix}.$$

have been frequently studied and possess many interesting properties.

Zehfuss, *Zeitschrift* VII., p. 439, states that they were discussed by Bessel in a correspondence with Hesse, about the year 1830. They have also been considered by Stern, *Crelle*, LXXIII. p. 374.

The peculiarity of the determinant D is that it admits of resolution into linear factors, viz. it is the product of the n factors

$$x_1 + \alpha x_2 + \alpha^2 x_3 + \ldots + \alpha^{n-1} x_n \quad \ldots\ldots\ldots\ldots\ldots\ldots (i).$$

when α is a root of the equation $z^n - 1 = 0$. See Spottiswoode, *Crelle*, LI., 375, or Baltzer, p. 100.

We have then

$$D = (x_1 + x_2 + \ldots + x_n)\,\Pi\,(x_1 + \alpha x_2 + \ldots + \alpha^{n-1}x_n)\ldots\text{(ii)},$$

where the product Π includes all factors of the form (i).

I propose in the present note to point out one or two properties of this determinant and to consider one or two special cases. The factor corresponding to $\alpha = 1$ is kept outside the product as being more convenient in applications of the formula.

If u be the logarithm of the determinant, we have

$$\frac{du}{dx_i} = \frac{1}{x_1 + x_2 + \ldots + x_n} + \Sigma \frac{\alpha^{i-1}}{x_1 + \alpha x_2 + \ldots + \alpha^{n-1}x_n} \ldots\ldots\text{(iii)}.$$

Thus we have

$$\left.\begin{aligned} \frac{du}{dx_1} + \frac{du}{dx_2} + \ldots + \frac{du}{dx_n} &= \frac{n}{x_1 + \alpha x_2 + \ldots + \alpha^{n-1}x_n} \\ \frac{du}{dx_1} + \alpha\frac{du}{dx_1} + \ldots + \alpha^{n-1}\frac{du}{dx_n} &= \frac{n}{x_1 + \beta x_2 + \ldots + \beta^{n-1}x_n} \end{aligned}\right\} \ldots\text{(iv)},$$

where β is that root of $z^n - 1 = 0$ such that $\alpha\beta = 1$.

Multiplying the equations (iv) together we see at once by aid of (ii), that if D' be the determinant formed from D by substituting $\frac{du}{dx_1}\ldots\frac{du}{dx_n}$ for $x_1\ldots x_n$, then

$$DD' = n^n.$$

If, for brevity, we denote the factors of D by $X_1\ldots X_n$, we see that

$$\begin{aligned} \frac{d^2u}{dx_i dx_k} &= -\frac{1}{(x_1 + x_2 + \ldots + x_n)^2} - \Sigma \frac{\alpha^{i-1}\alpha^{k-1}}{(x_1 + \alpha x_2 + \ldots + \alpha^{n-1}x_n)^2} \\ &= -\Sigma \frac{d\log X_r}{dx_i} \cdot \frac{d\log X_r}{dx_k}. \end{aligned}$$

Hence, if $H(u)$ be the Hessian of u,

$$H(u) = (-1)^n \left| \frac{d\log X_r}{dx_i} \right|^2,$$

or

$$H(u) = \frac{(-1)^n}{D^2} \left| \frac{dX_r}{dx_i} \right|^2.$$

Thus

$$D^2 H(u) = n^n (-1)^{\frac{1}{2}\{n(n-1)\}};$$

to which various other forms may be given.

Many interesting results can be obtained by giving special values to the quantities $x_1, x_2, \ldots x_n$.

For brevity, let us write

$$D = | x_1, x_2, \ldots x_n | .$$

(I) $D = | a, a+b, a+2b, \ldots a+(n-1)b |$

$$= \tfrac{1}{2}n\{2a+(n-1)b\}\Pi\left[\frac{a-\{a+(n-1)b\}\alpha^n}{1-\alpha} + \frac{b\alpha(1-\alpha^{n-1})}{(1-\alpha)^2}\right],$$

which since $\alpha^n = 1$ gives

$$D = \tfrac{1}{2}n\{2a+(n-1)b\}\,\Pi\left(\frac{-nb}{1-\alpha}\right)$$

$$= \frac{2a+(n-1)b}{2}(-nb)^{n-1}.$$

A special case of this is

$$| 1, 2 \ldots n | = \tfrac{1}{2}(n+1)(-n)^{n-1}.$$

(II) $$| 1^2, 2^2, \ldots n^2 | = (-1)^{n-1}\frac{(n+1)(2n+1)n^{n-2}}{12}\{(n+2)^n - n^n\}.$$

(III) $$D = | c_1, c_2, \ldots c_n | ,$$

where $c_1, c_2, \ldots c_n$ are the coefficients of the binomial expansion $(1+x)^{n-1}$. Then

$$D = 2^{n-1}, \text{ if } n \text{ is odd},$$

$$= 0, \quad \text{if } n \text{ is even}.$$

(IV) $D = | \cos a, \cos(a+b) \ldots \cos\{a+(n-1)b\} | .$

To effect the evaluation of this determinant, substitute for the cosines their exponential values

$$\cos a = \tfrac{1}{2}\left(u + \frac{1}{u}\right), \ \cos(a+b) = \tfrac{1}{2}\left(uv + \frac{1}{uv}\right), \ \&c.$$

Then D is the product of such factors as

$$\tfrac{1}{2}\left\{\left(u+\frac{1}{u}\right) + \left(uv+\frac{1}{uv}\right)\alpha + \ldots + \left(uv^{n-1} + \frac{1}{uv^{n-1}}\right)\alpha^{n-1}\right\}$$

$$= \tfrac{1}{2}\left\{u\,\frac{1-v^n}{1-v\alpha} + \frac{1}{u}\,\frac{1-v^{-n}}{1-v^{-1}\alpha}\right\}$$

$$= \frac{\cos\alpha - \cos(a+nb) - \alpha[\cos(a-b) - \cos\{a+(n-1)b\}]}{\alpha\left(\alpha + \frac{1}{\alpha} - 2\cos b\right)};$$

therefore

$$D=\frac{[\cos a-\cos(a+nb)]^n-[\cos(a-b)-\cos\{a+(n-1)b\}]^n}{2^n(1-\cos nb)}.$$

(V). In like manner

$$|\sin a, \sin(a+b)\ldots\sin\{a+(n-1)b\}|$$
$$=\frac{[\sin a-\sin(a+nb)]^n-[\sin(a-b)-\sin\{a+(n-1)b\}]^n}{2^n(1-\cos nb)}.$$

3. We are also able to solve the partial differential equation

$$\begin{vmatrix} D_1, & D_2, & \ldots D_n \\ D_n, & D_1, & \ldots D_{n-1} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ D_2, & D_3, & \ldots D_1 \end{vmatrix} u=0,$$

where $$D_r=\frac{d}{dx_r}.$$

Viz. the solution of this equation is the sum of the solutions of the different equations

$$\frac{du}{dx_1}+\alpha\frac{du}{dx_2}+\ldots+\alpha^{n-1}\frac{du}{dx_n}=0,$$

where $\alpha^n=1$. This is of Lagrange's form, and

$$u=f(x_2-\alpha x_1,\ x_3-\alpha^2 x_1,\ \ldots x_n-\alpha^{n-1}x_1).$$

Thus the solution of the given equation is

$$u=\Sigma f(x_2-\alpha x_1,\ x_3-\alpha^2 x_1,\ \ldots x_n-\alpha^{n-1}x_1),$$

when the summation extends to all values of α being roots of the equation $z^n-1=0$.

ON THE RELATIONS BETWEEN THE ANGLES OF FIVE CIRCLES IN A PLANE OR OF SIX SPHERES IN SPACE

By *Edouard Lucas.*

THE definition of the potency of a point with respect to a circle or a sphere, its expression in the system of Cartesian coordinates, and the most elementary properties of determinants, enable us to arrive immediately at a knowledge of the most general relations concerning the distances of points, angles, straight lines, and circles in a plane, and the angles of planes and spheres in space. The relations of the

distances and angles are included, to a great extent, in the following fundamental proposition.

The potencies of any point with respect to five circles in a plane or with respect to six spheres in space are connected by a linear and homogeneous equation in which the sum of the coefficients is zero.

Let $$x_i = x^2 + y^2 - 2a_i x - 2b_i y + c_i,$$

the first member of the equation of a circle in rectangular coordinates; we know that c_i and x_i represent respectively the potency of the origin and of any point M, coordinates x, y, with respect to this circle. Eliminating x, y and x^2+y^2 between the four equations obtained with $i = 1, 2, 3, 4$, we have the identity*

$$\begin{vmatrix} x_1, & a_1, & b_1, & 1 \\ x_2, & a_2, & b_2, & 1 \\ x_3, & a_3, & b_3, & 1 \\ x_4, & a_4, & b_4, & 1 \end{vmatrix} = \begin{vmatrix} c_1, & a_1, & b_1, & 1 \\ c_2, & a_2, & b_2, & 1 \\ c_3, & a_3, & b_3, & 1 \\ c_4, & a_4, & b_4, & 1 \end{vmatrix}.$$

Denote by l, m, n, p the coefficients of x_1, x_2, x_3, x_4 in the development of the determinant forming the first member, and by q the value of the second member, we have then

$$lx_1 + mx_2 + nx_3 + px_4 - q = 0 \quad \ldots\ldots\ldots\ldots (1);$$

but if we replace x_1, x_2, x_3, x_4 by unity or by the same number the first determinant vanishes, and we have

$$l + m + n + p = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

We thus obtain the following principle:

The potencies of a point with respect to four circles in a plane are connected by a linear equation, not homogeneous, in which the sum of the coefficients of the powers is zero.

If the four circles are orthogonal to a fifth, the centre of the latter has the same potency with respect to the four circles, whence $q = 0$, and consequently:

The potencies of a point with respect to four circles, orthogonal to the same circle, are connected by a linear and homogeneous equation in which the sum of the coefficients is zero.

To prove the fundamental theorem it suffices to consider a fifth circle, and we have, for example,

$$l'x_1 + m'x_2 + n'x_3 + p'x_5 - q' = 0 \ldots\ldots\ldots\ldots (3),$$
$$l' + m' + n' + p' = 0 \ldots\ldots\ldots\ldots (4).$$

* *Nouvelles Annales de Mathématiques*, Second Series. t. XV. p. 205.

Multiplying the two members of the equalities (1) and (2) by q', those of the equalities (3) and (4) by q, and subtracting, we have the demonstration of the theorem.

We define the *mutual potency* of two circles radii r_i and r_j, centres O_i and O_j, distance between the centres d_{ij}, to be A_{ij}, where

$$2r_ir_jA_{ij} = r_i^2 + r_j^2 - d_{ij}^2 \quad \ldots\ldots\ldots\ldots\ldots(5);$$

when the circles cut, A_{ij} represents the cosine of the angle of the two circles; we have also

$$d_{ii} = 0, \quad A_{ii} = 1, \quad A_{ij} = A_{ji}.$$

Now suppose the point M to be at the centre O_j of a circle of radius r_j; the potency of this point with respect to the circle of centre O_i is

$$x_i = d_{ij}^2 - r_i^2,$$

viz.
$$x_i = r_j^2 - 2r_ir_jA_{ij} \quad \ldots\ldots\ldots\ldots\ldots(6);$$

but, by the fundamental theorem,

$$lx_1 + mx_2 + nx_3 + px_4 + qx_5 = 0,$$
$$l \quad +m \quad +n \quad +p \quad +q \quad = 0;$$

and consequently comparing the three preceding formulæ,

$$lr_1r_jA_{1j} + mr_2r_jA_{2j} + nr_3r_jA_{3j} + pr_4r_jA_{4j} + qr_5r_jA_{5j} = 0.$$

Making j equal to 1, 2, 3, 4, 5, in the preceding equation we obtain five equations, which give, by the elimination of l, m, n, p, q, the determinant

$$\begin{vmatrix} r_1r_1A_{11}, & r_1r_2A_{12}, & r_1r_3A_{13}, & r_1r_4A_{14}, & r_1r_5A_{15} \\ r_2r_1A_{21}, & r_2r_2A_{22}, & r_2r_3A_{23}, & r_2r_4A_{24}, & r_2r_5A_{25} \\ r_3r_1A_{31}, & r_3r_2A_{32}, & r_3r_3A_{33}, & r_3r_4A_{34}, & r_3r_5A_{35} \\ r_4r_1A_{41}, & r_4r_2A_{42}, & r_4r_3A_{43}, & r_4r_4A_{44}, & r_4r_5A_{45} \\ r_5r_1A_{51}, & r_5r_2A_{52}, & r_5r_3A_{53}, & r_5r_4A_{54}, & r_5r_5A_{55} \end{vmatrix} = 0 \ldots\ldots(7).$$

If no one of the radii is zero, we can suppress all the radii; if one of them r_i is zero, we can replace $r_ir_jA_{ij}$ by $d^2_{ij} - r_j^2$, and $r_ir_iA_{ii}$ by 0; if two radii r_i and r_j are zero, we replace r_ir_j by d^2_{ij}. Thus, when the system of two circles is formed of a circle and a point, we replace in the relation (7) the mutual potency of the two circles by the potency of the point with regard to the circle; and if the system is formed of two points, we replace the mutual potency by the square of their distance.

If the radius r_i of one of the circles increases indefinitely, and if the circle becomes a straight line, divide by r_i and

replace A_{ij} by $\frac{\delta_j}{r_j}$ or by the cosine of the angle between the straight line and the circle, when the straight line and circle cut one another. Thus, when the system of two circles is formed of a circle and a straight line, we replace in the relation (7) the mutual potency by the quotient of the distance of the centre to the straight line by the radius; we see also that when the system of two circles is formed of a point and a straight line, we replace the mutual potency by the distance of the point from the straight line.

In the system formed by a circle and a straight line which passes to an indefinite distance, we replace the mutual potency by the reciprocal of the radius.

In the system formed by a point and the straight line at infinity, we replace the mutual potency by unity; in the system formed by two straight lines we replace the mutual potency by the cosine of their angle; if one of the straight lines passes to an infinite distance, we replace the mutual potency by zero.

Consequently, applying, as has been said, the notion of the mutual potency of the points, of the straight lines, of the circles and of the straight lines at infinity, we have the following theorem:

The determinant formed by the mutual potencies of any points, straight lines or circles in a plane, whose number is at least equal to *five*, is identically zero.

We have evidently the same theorem in space for any six points, planes or spheres, or for any one or a greater number of elements.

The following are some corollaries from this important theorem:

1. If the circle O_5 is a point on the circle O_4, we have the formula

$$\begin{vmatrix} A_{11}, & A_{12}, & A_{13}, & A_{14}, & \frac{x_1}{r_1} \\ A_{21}, & A_{22}, & A_{23}, & A_{24}, & \frac{x_2}{r_2} \\ A_{31}, & A_{32}, & A_{33}, & A_{34}, & \frac{x_3}{r_3} \\ A_{41}, & A_{42}, & A_{43}, & A_{44}, & 0 \\ \frac{x_1}{r_1}, & \frac{x_2}{r_2}, & \frac{x_3}{r_3}, & 0, & 1 \end{vmatrix} = 0.$$

This is the equation of the system of two circles cutting three given circles $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ at given angles.

2. We obtain, besides, the four groups of these circles conformably to the rule of signs given in the *Nouvelle Correspondance Mathématique.*

3. If $A_{14} = A_{24} = A_{34} = \pm 1$, we have the equation of the system of two circles touching internally or externally three given circles.

4. If $A_{14} = A_{24} = A_{34} = 0$, we obtain the equation of the circle orthogonal to three given circles.

5. If the given circles reduce to three points, we obtain the equation of the circumscribed circle.

6. If the circles become three straight lines, we obtain the equation of the inscribed and escribed circles.

7. If, besides, $A_{14} = A_{24} = A_{34} = 0$, we obtain the equation of the double straight line at infinity.

8. Suppose now that the circle O_5 becomes the straight line at infinity, we then have the formula

$$\begin{vmatrix} A_{11}, & A_{12}, & A_{13}, & A_{14}, & \frac{1}{r_1} \\ A_{21}, & A_{22}, & A_{23}, & A_{24}, & \frac{1}{r_2} \\ A_{31}, & A_{32}, & A_{33}, & A_{34}, & \frac{1}{r_3} \\ A_{41}, & A_{42}, & A_{43}, & A_{44}, & \frac{1}{r_4} \\ \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & \frac{1}{r_4}, & 0 \end{vmatrix} = 0;$$

this is the equation giving the radii of the three circles which cut three given straight lines or circles at given angles.

9. If $A_{14} = A_{24} = A_{34} = \pm 1$, we find again the formula giving the radii of the system of circles touching three given circles. (Bauer, *Journal de Schlömilch*, t. v.).

10. If $A_{14} = A_{24} = A_{34} = 0$, we find the radius of the circle orthogonal to three given circles.

11. If $r_1 = r_2 = r_3 = 1$, we obtain the radius of the circle circumscribing the triangle, given by the three points O_1, O_2, O_3, and consequently the area of the triangle.

12. If $r_1 = r_2 = r_3 = r_4 = 1$, we obtain the relation given by Cayley between the distances of four points in a plane (*Cambridge Journal*, t. II.).

13. If the elements O_4 and O_5 become two straight lines at infinity, we obtain the relation between the cosines of the angles of three directions in a plane.

14. If the elements O_1, O_2, O_3 represent the origin of the coordinates and the two axes, O_4 any circle whatever, and O_5 a point on this circle, we obtain the equation of the circle O_4 in oblique coordinates.

15. If the element O_5 is replaced by the straight line at infinity, we obtain the radius of a circle with given centre, which cuts at given angles the two axes of coordinates, &c.

These considerations are also applicable very easily to space.

Paris, *December*, 1877.

MATHEMATICAL NOTES.

Proof of the Theorem in Kinematics, vol. VII. p. 190.

Let P, P' be two points on the moving plane, and (P), (P') the areas described by them.

Let $PP' = r$, and let PP' make N revolutions.

Let n be the total movement of P' perpendicular to $P'P$.

Then $(P) - (P') = nr + N\pi r^2$.

Take P' as origin and the direction of $P'P$ in which n is a maximum $(= n')$ as initial line.

Then $$n = n' \cos\theta.$$

Thus $$(P) - (P') = n'r\cos\theta + N\pi r^2,$$

the equation to a family of concentric circles.

Transform to centre, then

$$(P) = N\pi(r^2 - a^2)$$

where $a =$ radius of circle corresponding to $(P) = 0$.

A. B. KEMPE.

Fluid Motion in a Rotating Semicircular Cylinder.

It may be useful to put on record the solution of this simple case of fluid motion. Take the centre of the semicircle as origin and one of the bounding radii as initial line.

Let a be the radius and ω the angular velocity. Expand ϕ, the velocity potential in the series of harmonics

$$\phi = \Sigma \left\{ A_n \left(\frac{r}{a}\right)^n + B_n \left(\frac{a}{r}\right)^n \right\} \cos(n\theta + \alpha).$$

The conditions to be satisfied by ϕ are (1) it must be finite everywhere, (2) $\frac{d\phi}{rd\theta} = \omega r$ when $\theta = 0$ or $= \pi$ and (3) $\frac{d\phi}{dr} = 0$ when $r = a$ for all values of θ from $\theta = 0$ to $\theta = \pi$.

Condition (1) is satified by putting $B = 0$.

..... (2)..... $\alpha = 0$, except when $n=2$, when there must be a term $\frac{1}{2}\omega r^2 \sin 2\theta$. Thus

$$\phi = \tfrac{1}{2}\omega r^2 \sin 2\theta + \Sigma_1^\infty A_n \left(\frac{r}{a}\right)^n \cos n\theta.$$

To satisfy the third condition

$$\omega a \sin 2\theta + \Sigma_1^\infty \frac{n}{a} A_n \cos n\theta = 0 \text{ from } \theta = 0 \text{ to } \theta = \pi.$$

Therefore n is integral and by applying Fourier's theorem we shall find that $A_{2m} = 0$,

$$A_{2m+1} = \frac{\omega a^2}{\pi} \left\{ \frac{1}{2m-1} - \frac{2}{2m+1} + \frac{1}{2m+3} \right\}.$$

Write $2\cos n\theta = e^{n\theta i} + e^{-n\theta i}$, and put $\frac{r}{a} e^{\theta i} = x$. We shall require to sum the series

$$(-1 - \tfrac{2}{1} + \tfrac{1}{3})x + (1 - \tfrac{2}{3} + \tfrac{1}{5})x^3 + \ldots + \left(\frac{1}{2m-1} - \frac{2}{2m+1} + \frac{1}{2m+3}\right) x^{2m+1} + \ldots.$$

Now $\log(1+x) - \log(1-x) = 2\left\{x + \frac{x^3}{3} + \ldots + \frac{x^{2m+1}}{2m+1} + \ldots\right\}$;

therefore $$\Sigma_0^\infty \frac{1}{2m+3} x^{2m+1} = \frac{1}{2x^2} \log \frac{1+x}{1-x} - \frac{1}{x}$$

$$\Sigma_0^\infty \frac{1}{2m-1} x^{2m+1} = \frac{x^2}{2} \log \frac{1+x}{1-x} - x;$$

therefore

$$\Sigma_0^\infty \left\{ \frac{1}{2m-1} - \frac{2}{2m+1} + \frac{1}{2m+3} \right\} x^{2m+1} = \tfrac{1}{2}\left(x^2 + \frac{1}{x^2} - 2\right) \log \frac{1+x}{1-x} - \left(x + \frac{1}{x}\right).$$

Hence

$$\phi = \tfrac{1}{2}\omega r^2 \sin 2\theta + \frac{\omega a^2}{2\pi}\left[\tfrac{1}{2}\left\{\left(\frac{r}{a}e^{\theta i}\right)^2 + \left(\frac{a}{r}e^{-\theta i}\right)^2 - 2\right\}\log\frac{a+re^{\theta i}}{a-re^{\theta i}}\right.$$
$$\left. - \left(\frac{r}{a}e^{\theta i} + \frac{a}{r}e^{-\theta i}\right) + \tfrac{1}{2}\left\{\left(\frac{r}{a}e^{-\theta i}\right)^2 + \left(\frac{a}{r}e^{\theta i}\right)^2 - 2\right\}\log\frac{a+re^{-\theta i}}{a-re^{-\theta i}}\right.$$
$$\left. - \left(\frac{re^{-\theta i}}{a} + \frac{ae^{\theta i}}{r}\right)\right]$$

$$= \tfrac{1}{2}\omega r^2 \sin 2\theta + \frac{\omega a^2}{2\pi}\left[\tfrac{1}{2}\left\{\left(\frac{r^2}{a^2} + \frac{a^2}{r^2}\right)\cos 2\theta - 2\right\}\log\frac{(a+re^{\theta i})(a+re^{-\theta i})}{(a-re^{\theta i})(a-re^{-\theta i})}\right.$$
$$+ \tfrac{1}{2}\left(\frac{r^2}{a^2} - \frac{a^2}{r^2}\right) i \sin 2\theta \log\frac{(a+re^{\theta i})(a-re^{-\theta i})}{(a+re^{-\theta})(a-re^{\theta i})}$$
$$\left. - 2\left(\frac{r}{a} + \frac{a}{r}\right)\cos\theta\right]$$

$$= \tfrac{1}{2}\omega r^2 \sin 2\theta + \frac{\omega a^2}{4\pi}\left[\left\{\left(\frac{r^2}{a^2} + \frac{a^2}{r^2}\right)\cos 2\theta - 2\right\}\log\frac{r^2 + 2ar\cos\theta + a^2}{r^2 - 2ar\cos\theta + a^2}\right.$$
$$- 2\left(\frac{r^2}{a^2} - \frac{a^2}{r^2}\right)\sin 2\theta \tan^{-1}\frac{2ar\sin\theta}{a^2 - r^2}$$
$$\left. - 4\left(\frac{r}{a} + \frac{a}{r}\right)\cos\theta\right].$$

This result was set to be verified in a St. John's College examination paper in June, 1876; but was first obtained in the above manner. It is interesting as a solution obtained by a series which sums to a finite expression.

W. M. Hicks.

On the Caustic by Refraction of a Circle for Parallel Rays.

If the radius of the circle be taken as unity, and if ϕ, ϕ' be the angles of incidence and refraction, the incident rays being parallel to the axis to x, then the equation of the refracted ray is

$$(y\cos\phi - x\sin\phi)\cos\phi' + (y\sin\phi + x\cos\phi - 1)\sin\phi' = 0 \ldots \text{(i)},$$

where $\sin\phi' = k\sin\phi$ and $\cos\phi' = \sqrt{(1 - k^2\sin^2\phi)}$, k being the reciprocal of μ, the index of refraction. Now put $\phi = \operatorname{am} u$, and (i) becomes

$$(y\,\operatorname{cn} u - x\,\operatorname{sn} u)\operatorname{dn} u + (y\,\operatorname{sn} u + x\,\operatorname{cn} u - 1)\,k\,\operatorname{sn} u = 0,$$

viz. $$\operatorname{sn} u\,(k\operatorname{cn} u - \operatorname{dn} u)\,x + (k\operatorname{sn}^2 u + \operatorname{cn} u\operatorname{dn} u)\,y = k\operatorname{sn} u.$$

Multiply throughout by $k\,\mathrm{cn}\,u + \mathrm{dn}\,u$, and this equation becomes

$$-k'^2\,\mathrm{sn}\,u\,x + (\mathrm{cn}\,u + k\,\mathrm{dn}\,u)\,y = k\,\mathrm{sn}\,u\,(k\,\mathrm{cn}\,u + \mathrm{dn}\,u),$$

and we have to find the envelope of this line.

Divide by $\mathrm{sn}\,u$, giving

$$-k'^2 x + \frac{\mathrm{cn}\,u + k\,\mathrm{dn}\,u}{\mathrm{sn}\,u}\,y = k\,(k\,\mathrm{cn}\,u + \mathrm{dn}\,u) \quad \ldots\ldots(\mathrm{ii});$$

differentiate with regard to u, and we have

$$-\frac{\mathrm{dn}\,u + k\,\mathrm{cn}\,u}{\mathrm{sn}^2 u}\,y = -k^2\,\mathrm{sn}\,u\,(\mathrm{dn}\,u + k\,\mathrm{cn}\,u),$$

whence $$y = k^2\,\mathrm{sn}^3 u,$$

and substituting in (ii)

$$-k'^2 x = k\,(k\,\mathrm{cn}\,u + \mathrm{dn}\,u) - k^2\,\mathrm{sn}^2 u\,(\mathrm{cn}\,u + k\,\mathrm{dn}\,u)$$
$$= k^2\,\mathrm{cn}^3 u + k\,\mathrm{dn}^3 u.$$

whence the equation to the caustic is

$$k'^2 x = -k^2\{1 - k^{-\frac{4}{3}} y^{\frac{2}{3}}\}^{\frac{3}{2}} - k\{1 - k^{\frac{2}{3}} y^{\frac{2}{3}}\}^{\frac{3}{2}};$$

that is $$(1-\mu^2)\,x = (1 - \mu^{\frac{4}{3}} y^{\frac{2}{3}})^{\frac{3}{2}} + \mu\,(1 - \mu^{-\frac{2}{3}} y^{\frac{2}{3}})^{\frac{3}{2}}.$$

The above investigation of the caustic is the same as that given by Professor Cayley in his "Memoir on Caustics," (Phil. Trans., 1857, pp. 281, 282) except that the envelope of (i) is there found by differentiation with regard to ϕ, and the elimination of ϕ is difficult, occupying a page of work. Now that elliptic functions are coming into general use, it seems as proper in a piece of algebra to replace $\sin\phi$, $\cos\phi$, $\sin\sqrt{(1 - k^2\sin^2\phi)}$ by $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$ (or, as is most convenient, in actual work by s, c, d) as to replace α, $\sqrt{(1-\alpha^2)}$ by $\sin\alpha$, $\cos\alpha$; and it is clear that in optical questions, where we have $\sin\phi$, $\cos\phi$, $\sin\phi'$, $\cos\phi'$ involved, the change may often be convenient. Prof. Cayley himself puts k for μ^{-1}, so that in the memoir $\sqrt{(1 - k^2\sin^2\phi)}$ is written for $\cos\phi'$, which indicates that the elliptic function analogy was remarked by him.

J. W. L. GLAISHER.

An Algebraical Identity.

Let a, b, c, f, g, h be the differences of four quantities α, β, γ, δ, say

$$a, b, c, f, g, h = \beta - \gamma,\ \gamma - \alpha,\ \alpha - \beta,\ \alpha - \delta,\ \beta - \delta,\ \gamma - \delta,$$

then

$$\begin{aligned} . \quad h - g + a &= 0, \\ -h \quad . + f + b &= 0, \\ g - f \quad . + c &= 0, \\ -a - b - c \quad . &= 0. \end{aligned}$$

Now Cauchy's identity

$$(a+b)^7 - a^7 - b^7 = 7ab\,(a+b)\,(a^2+ab+b^2)^2,$$

putting therein $a+b=-c$, becomes

$$a^7 + b^7 + c^7 = 7abc\,(bc+ca+ab)^2;$$

hence we have

$$\begin{aligned} . \quad h^7 - g^7 + a^7 &= -7agh\,(-ga+ah-hg)^2, \\ -h^7 \quad . + f^7 + b^7 &= -7bhf\,(-hb+bf-fh)^2, \\ g^7 - f^7 \quad . + c^7 &= -7cfg\,(-fc+cg-gf)^2, \\ -a^7 - b^7 - c^7 \qquad &= -7abc\,(\ bc+ca+ab)^2; \end{aligned}$$

whence, adding,

$$agh\,(-ga+ah-hg)^2 + bhf(-hb+bf-fh)^2 + cfg\,(-fc+cg-gf)^2 + abc\,(bc+ca+ab)^2 = 0,$$

or, as this may also be written,

$$agh\,(g^2+h^2+a^2)^2 + bhf(h^2+f^2+b^2)^2 + cfg\,(f^2+g^2+c^2)^2 + abc\,(a^2+b^2+c^2)^2 = 0,$$

an identity if a, b, c, f, g, h denote their values in terms of $\alpha, \beta, \gamma, \delta$. A. CAYLEY.

[1°. The identity may also be obtained by equating to zero the terms of the seventh order in the trigonometrical identity

$$\sin(\beta-\gamma)\sin(\beta-\delta)\sin(\gamma-\delta) + \sin(\gamma-\alpha)\sin(\gamma-\delta)\sin(\alpha-\delta) + \sin(\alpha-\beta)\sin(\alpha-\delta)\sin(\beta-\delta) + \sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta) = 0;$$

that is, in

$$\sin a \sin g \sin h + \sin b \sin h \sin f + \sin c \sin f \sin g + \sin a \sin b \sin c = 0.$$

For the terms of the seventh order in the expansion of $\sin a \sin b \sin c$ are

$$-\tfrac{1}{5\,6\,0} abc\,(3a^4 + 3b^4 + 3c^4 + 10b^2c^2 + 10c^2a^2 + 10a^2b^2) \ldots (1),$$

and if $a+b+c=0$, we have

$$a^4 + b^4 + c^4 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2,$$

and also

$$(bc+ca+ab)^2 = b^2c^2 + c^2a^2 + a^2b^2.$$

Thus (1)

$$= \tfrac{8}{360}abc\,(a^4+b^4+c^4) = \tfrac{16}{360}abc\,(b^2c^2+c^2a^2+a^2b^2)$$

$$= \tfrac{16}{360}abc\,(bc+ca+ab)^2,$$

so that the identity is

$$agh\,(g^4+h^4+a^4) + bhf\,(h^4+f^4+b^4)$$

$$+ cfg\,(f^4+g^4+c^4) + abc\,(a^4+b^4+c^4) = 0 \ldots\ldots\ldots (2),$$

or

$$agh\,(-ga+ah-hg)^2 + \&c. = 0,$$

or

$$agh\,(g^2+h^2+a^2)^2 + \&c. = 0.$$

2°. It may be remarked that, by a known theorem,

$$(x+y)^n - x^n - y^n = nxy\,(x+y)$$

$$\times \left\{(x+y)^{n-3} - \frac{n-3}{2}\,xy(x+y)^{n-5} + \frac{(n-4)(n-5)}{2.3}\,(xy)^2(x+y)^{n-7} - \&c.\right\},$$

whence, if $a+b+c=0$, and n be uneven

$$a^n + b^n + c^n = nabc\,\{a^{n-3} - \tfrac{1}{2}\,(n-3)\,bca^{n-5} + \&c.\},$$

writing the two similar expressions obtained by interchanging a, b, c, we have by addition

$$a^n + b^n + c^n = \tfrac{1}{3}nabc$$

$$\times \{a^{n-3} + b^{n-3} + c^{n-3} - \tfrac{1}{2}\,(n-3)\,abc\,(a^{n-6} + b^{n-6} + c^{n-6}) + \&c.\},$$

whence we can write down an identity of the n^{th} order, that includes (2).

3°. By Cauchy's theorem if n be of the form $6m+1$, then $(x+y)^n - x^n - y^n$ is divisible by $(x^2+xy+y^2)^2$, so that

$$(x+y)^7 - x^7 - y^7 = 7xy\,(x+y)\,(x^2+xy+y^2)^2.$$

By actual division, I find that

$$(x+y)^{13} - x^{13} - y^{13} = 13xy\,(x+y)\,(x^2+xy+y^2)^2$$

$$\times (x^6 + 3x^5y + 8x^4y^2 + 11x^3y^3 + 8x^2y^4 + 3xy^5 + y^6),$$

and the last factor

$$= (x^2+xy+y^2)^3 + 2x^2y^2\,(x+y)^2.$$

I find also that

$$(x+y)^{11} - x^{11} - y^{11} = 11xy\,(x+y)\,(x^2+xy+y^2)$$

$$\times \{(x^2+xy+y^2)^3 + x^2y^2\,(x+y)^2\}.$$

J. W. L. GLAISHER].

On Certain Sums of Squares.

The $\frac{1}{2}n(n-1)$ squares $\Sigma(a_i - a_j)^2$ formed by adding the squares of every pair of the quantities $a_1, a_2, \ldots a_n$ may, if n be uneven, be expressed as the sum of n squares in the form

$$(a_1 + a_2c_1 + a_3c_2 \ldots + a_nc_{n-1})^2 + (a_2s_1 \ldots + a_ns_{n-1})^2$$
$$+ (a_1 + a_2c_2 + a_3c_4 \ldots + a_nc_{2n-2})^2 + (a_2s_2 \ldots + a_ns_{2n-2})^2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ (a_1 + a_2c_{\frac{1}{2}(n-1)} \ldots + a_nc_{\frac{1}{2}(n-1)^2})^2 + (a_2s_{\frac{1}{2}(n-1)} \ldots + a_ns_{\frac{1}{2}(n-1)^2})^2,$$

where c_m stands for $\cos\frac{2m\pi}{n}$, and s_m for $\sin\frac{2m\pi}{n}$.

For, in this expression, the coefficient of a_i^2 is $\frac{1}{2}(n-1)$ and the coefficient of a_ia_j is equal to

$$2\cos\frac{2(i-j)\pi}{n} + 2\cos\frac{4(i-j)\pi}{n} \ldots + 2\cos\frac{(n-1)(i-j)\pi}{n}$$
$$= \omega^{i-j} + \omega^{2(i-j)} \ldots + \omega^{(n-1)(i-j)} = -1,$$

where $$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}.$$

Thus the whole expression

$$= \tfrac{1}{2}(n-1)\Sigma a_i^2 - \Sigma a_ia_j = \tfrac{1}{2}\Sigma(a_i - a_j)^2.$$

If n be even, then the expression

$$(a_1 + a_2c_1 + a_3c_2 \ldots + a_nc_{n-1})^2 + (a_2s_1 \ldots + a_ns_{n-1})^2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ (a_1 + a_2c_{\frac{1}{2}n-1} \ldots + a_nc_{(\frac{1}{2}n-1)(n-1)})^2 + (a_2s_{\frac{1}{2}n-1} \ldots + a_ns_{(\frac{1}{2}n-1)(n-1)})^2$$
$$= \Sigma(a_i - a_j)^2,$$

where $i-j$ must be even. For the coefficient of a_ia_j is equal to the sum of the $(i-j)^{\text{th}}$ powers of all the imaginary roots of the equations $x^n - 1 = 0$, so that if $i-j$ be uneven, the coefficient is zero and if $i-j$ be even it $= -2$.

The simplest examples are $(n=3)$

$$(a_1 - a_2)^2 + (a_1 - a_3)^2 + (a_2 - a_3)^2$$
$$= 2\{(a_1 + a_2\cos\tfrac{2}{3}\pi + a_3\cos\tfrac{4}{3}\pi)^2 + (a_2\sin\tfrac{2}{3}\pi + a_3\sin\tfrac{4}{3}\pi)^2\},$$

and $(n=4)$

$$(a_1 - a_3)^2 + (a_2 - a_4)^2$$
$$= (a_1 + a_2\cos\tfrac{1}{2}\pi + a_3\cos\pi + a_4\cos\tfrac{3}{2}\pi)^2 + (a_2\sin\tfrac{1}{2}\pi + a_3\sin\pi + a_4\sin\tfrac{3}{2}\pi)^2.$$

J. W. L. GLAISHER.

Centres of Pressure.

The centres of pressure of (1) a parallelogram with side in surface, (2) a triangle with an angular point in the surface and the opposite side horizontal may be deduced from that of a triangle with one side in the surface by two methods. To make the note complete we will find:—

(i). Centre of pressure of triangle with one side in the surface. Divide the triangle up into indefinitely thin horizontal strips of equal breadth. Let DE, FG (fig. 1) be two such strips at equal distances from AB and C respectively. Bisect AB in H, join CH meeting DE in M and FG in N.

The pressure at any point of DE: that at any point of $FG :: AD : AF :: CF : CD :: FG : DE$; therefore pressure on strip DE = pressure on strip FG, and centres of pressure of these strips are at M and N respectively; therefore centre of pressure of these, or of any two such strips, is at middle point O of CH. But such strips comprise the whole triangle, therefore centre of pressure of the triangle is at middle point of CH.

(ii). To deduce the centre of pressure of (1) the parallelogram and (2) the triangle. $ABCD$ (fig. 2) is a parallelogram with AB in surface, E and F are middle points of AB, CD respectively, join BD, DE, BF. Centre of pressure of ABD is in DE; centre of pressure of BDC is in BF; pressure on BDC = twice that on ABD, because the areas of the 2 triangles are equal, but the depth of centre of gravity of one is twice that of the other. Therefore centre of pressure of $ABCD$ is in a line parallel to DE and BF, and one third of the interval distant from BF, but it is also in EF; it is therefore at G, such that $EG = 2GF$. (This does not depend upon the actual position of centre of pressure of ABD).

Now centre of pressure of ABD is at H, the middle point of DE. Join DG and produce to meet BF in K; then K is the centre of pressure of BDC, and

$$FK : HE :: FG : GE :: 1 : 2;$$

therefore

$$FK = \tfrac{1}{2}HE = \tfrac{1}{4}BF.$$

(iii) We may deduce these results by another method from (i). Let ABC (fig 3) be a triangle, A being in the surface; produce BC to meet the surface in D. Bisect AD in E. Then centres of pressure of triangles ABD, ACD lie on BE, CE respectively, and at their middle points. The line joining these will therefore pass through centre of pressure

of the triangle ABC, which is their difference. This line is easily seen to be at one quarter of the distance from BD towards A. Let BC become horizontal, D therefore goes off to infinity. Then centre of pressure of each strip parallel to BC in triangle BBC will be at the middle point of it; therefore centre of pressure is three quarters of distance from A to middle point of BC.

(iv). Through G draw HGK parallel to AC, meeting AD, CD respectively in H, K. Then $HG : GK :: 2 : 1$, and when D becomes infinitely distant this will give the distance below the surface of the centre of pressure of any parallelogram having a side in the surface to be two-thirds of the distance between the two horizontal sides.

(v). If the lines we have supposed in the surface be kept horizontal but covered with liquid to a depth h, we must find the centre of pressure by uniting with the pressure already found (acting at its centre of pressure) a *uniform* pressure due to depth h (therefore acting at centre of gravity of area considered).

(vi). For example: to find the centre of gravity of a narrow straight strip immersed in liquid. If one end of the strip were in the surface, the pressure would be in magnitude the same as that due to a depth $\frac{1}{2}(d-d_1)$ over the area, and acts at two-thirds of the distance down the strip from the highest point of it [d, d_1 are the real depths of the two ends]. With this then we must combine a pressure due to depth d_1 over the surface of the strip and acting at its middle point.

Let AB be the strip (fig. 4), trisect it in C and D, and bisect it in O.

The force at O may be divided into two equal parts and supposed to act at C and D. We have, therefore, at C and D forces proportional to $\frac{1}{2}(d-d_1)+\frac{1}{2}d_1$ and $\frac{1}{2}d_1$ respectively, or proportional to d and d_1 the centre of pressure is therefore at a point P in CD, such that $CP : PD ::$ depth of B : depth of A.

(vii). To find the centre of pressure of any triangle anywhere in the fluid.

Firstly, let the point A of the triangle ABC be in the surface. Then, combining results of (iii) and (vi), we see that the centre of pressure is three-quarters of distance from A to a point P in BC, such that if b, c, be points of trisection of BC, $bP : Pc ::$ depth of C : depth of B.

Secondly, if A be not in the surface we must find the centre of two parallel forces, one at P and the other at the centre of gravity of the triangle and respectively proportional to the depth of that centre of gravity below A and the depth of A below the surface.

T. C. LEWIS.

June, 1878.

On the Deformation of a Model of a Hyperboloid.

The following is a solution of Mr. Greenhill's problem set in the Senate-house Examination, January 14, 1878.

"Prove that, if a model of a hyperboloid of one sheet be constructed of rods representing the generating lines, jointed at the points of crossing; then if the model be deformed it will assume the form of a confocal hyperboloid, and prove that the trajectory of a point on the model will be orthogonal to the system of confocal hyperboloids."

Let (x_1, y_1, z_1), (x_2, y_2, z_2) be points on the generating line of $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, then

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - \frac{x_1^2}{c^2} = 1,$$

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} - \frac{z_2^2}{c^2} = 1,$$

$$\frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2} - \frac{z_1 z_2}{c^2} = 1,$$

or what is the same thing, if

$$\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c} = p_1, q_1, r_1; \quad \frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c} = p_2, q_2, r_2;$$

then

$$p_1^2 + q_1^2 - r_1^2 = 1,$$
$$p_2^2 + q_2^2 - r_2^2 = 1,$$
$$p_1 p_2 + q_1 q_2 - r_1 r_2 = 1.$$

Similarly if (ξ_2, η_2, ζ_2), (ξ_2, η_2, ζ_2) be points on generating line of $\frac{\xi^2}{\alpha^2} + \frac{\gamma^2}{\beta^2} - \frac{\zeta^2}{\gamma^2} = 1$, and if

$$\frac{\xi_1}{\alpha}, \frac{\eta_1}{\beta}, \frac{\zeta_1}{\gamma} = p_1, q_1, r_1; \quad \frac{\xi_2}{\alpha}, \frac{\eta_2}{\beta}, \frac{\zeta_2}{\gamma} = p_2, q_2, r_2,$$

then

$$p_1^2 + q_1^2 - r_1^2 = 1,$$
$$p_2^2 + q_1^2 - r_2^2 = 1,$$
$$p_1 p_2 + q_1 q_2 - r_1 r_2 = 1.$$

Hence if (x_1, y_1, z_1), (ξ_1, η_1, ζ_1) be corresponding points on the two surfaces, that is if

$$\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c} = \frac{\xi_1}{\alpha}, \frac{\eta_1}{\beta}, \frac{\zeta_1}{\gamma}, = p_1, q_1, r_1,$$

and similarly (x_2, y_2, z_2), (ξ_2, η_2, ζ_2) are corresponding points, that is if

$$\frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c} = \frac{\xi_2}{\alpha}, \frac{\eta_2}{\beta}, \frac{\zeta_2}{\gamma} = p_2, q_2, r_2;$$

then we have, as before, the system of three equations

$$p_1^2 + q_1^2 - r_1^2 = 1,$$
$$p_2^2 + q_2^2 - r_2^2 = 1,$$
$$p_1 p_2 + q_1 q_2 - r_1 r_2 = 1.$$

Then if the two surfaces are confocal, that is if

$$\alpha^2, \beta^2, -\gamma^2 = a^2 + h, b^2 + h, -c^2 + h,$$

we shall have

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = (\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \xi_2)^2.$$

For this equation is

$$a^2 (p_1 - p_2)^2 + b^2 (q_1 - q_2)^2 + c^2 (r_1 - r_2)^2 = \alpha^2 (p_1 - p_2)^2 + \beta^2 (q_1 - q_2)^2 + \gamma^2 (r_1 - r_2)^2,$$

that is $$(p_1 - p_2)^2 + (q_1 - q_2)^2 - (r_1 - r_2)^2 = 0,$$

an equation which is obviously true in virtue of the above system of three equations.

Hence, if on confocal surfaces

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad \frac{\xi^2}{a^2 + h} + \frac{\eta^2}{b^2 + h} - \frac{z^2}{c^2 - h} = 1,$$

we take on the first two points P_1, P_2, and on the second the corresponding points Q_1, Q_2; then P_1, P_2 being on a generating line of the first surface, Q_1, Q_2 will be on a generating line of the second surface, and P_1P_2 will be $= Q_1Q_2$, and the same is evidently true for the quadrilaterals $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$ where P_1P_2, P_2P_3, P_3P_4, P_4P_1 are generating lines on the first surface, and therefore Q_1Q_2, Q_2Q_3, Q_3Q_4, Q_4Q_1 are generating lines on the second surface, which proves the theorem.

A. CAYLEY.

ON A CLASS OF ALGEBRAICAL IDENTITIES.

By *J. W. L. Glaisher.*

§1. CAUCHY'S theorem referred to on pp. 46, 47 is that $(x+y)^n - x^n - y^n$ is divisible by x^2+xy+y^2 if n be of the form $6m \pm 1$, and by $(x^2+xy+y^2)^2$ if n be of the form $6m+1$. The expression is also divisible by $xy(x+y)$ when n is uneven. The complete resolutions for $n=11$ and $n=13$ are given on p. 47, and working out the resolution for $n=9$, we have the remarkable series of identities

$$(x+y)^3 - x^3 - y^3 = 3xy(x+y),$$
$$(x+y)^5 - x^5 - y^5 = 5xy(x+y)(x^2+xy+y^2),$$
$$(x+y)^7 - x^7 - y^7 = 7xy(x+y)(x^2+xy+y^2)^2,$$
$$(x+y)^9 - x^9 - y^9 = 3xy(x+y)\{3(x^2+xy+y^2)^3 + x^2y^2(x+y)^2\},$$
$$(x+y)^{11} - x^{11} - y^{11} = 11xy(x+y)(x^2+xy+y^2) \times \{(x^2+xy+y^2)^3 + x^2y^2(x+y)^2\},$$
$$(x+y)^{13} - x^{13} - y^{13} = 13xy(x+y)(x^2+xy+y^2)^2 \times \{(x^2+xy+y^2)^3 + 2x^2y^2(x+y)^2\},$$

§2. These formulæ may be put in a symmetrical form by taking $x=c-b$, $y=a-c$, whence $x+y=a-b$, and n being uneven

$$(x+y)^n - x^n - y^n = (b-c)^n + (c-a)^n + a-b)^n.$$

Also
$$xy(x+y) = (b-c)(c-a)(a-b) = \tfrac{1}{8}\{(b-c)^3 + (c-a)^3 + (a-b)^3\},$$

and
$$x^2+xy+y^2 = \tfrac{1}{2}\{(b-c)^2 + (c-a)^2 + (a-b)^2\}.$$

Now, let
$$\frac{1}{n}\{(b-c)^n + (c-a)^n + (a-b)^n\}$$

be denoted by P_n; then

$$xy(x+y) = P_3, \quad x^2+xy+y^2 = P_2,$$

and it is readily proved that
$$P_4 = \tfrac{1}{2}P_2^2 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Cauchy's theorem then shows that, ignoring numerical factors and considering only algebraical divisibility, P_n is divisible by P_3, by P_3 and P_2, or by P_3, P_2 and P_4 according as n is of the form $6m+3$, $6m+5$ or $6m+1$.* In the last case P_n is divisible by $P_3P_2^2$, and therefore by P_3P_4 by (1), but of course P_n is not divisible by the product $P_2P_3P_4$.

* I have enunciated in this form Cauchy's theorem in the *Quarterly Journal of Mathematics*, vol. XV. p. 366 (1878).

The formulæ of the last section, when transformed, as above, become

$$\begin{aligned}
P_5 &= P_2P_3,\\
P_7 &= 2P_3P_4,\\
P_9 &= P_3(P_2^3+\tfrac{1}{3}P_3^2),\\
P_{11} &= P_2P_3(P_2^3+P_3^2),\\
P_{13} &= 2P_3P_4(P_2^3+2P_3^2).
\end{aligned}$$

The last two equations may also be written

$$\begin{aligned}
P_{11} &= P_5(P_2^3+P_3^2),\\
P_{13} &= P_7(P_2^3+2P_3^2),
\end{aligned}$$

so that $\frac{P_9}{P_3}$, $\frac{P_{11}}{P_5}$, $\frac{P_{13}}{P_7}$ are connected by a linear equation, which is easily found to be

$$2\frac{P_{13}}{P_7}-5\frac{P_{11}}{P_5}+3\frac{P_9}{P_3}=0.$$

§ 3. Taking as in Prof. Cayley's identity (pp. 45, 46), a, b, c, f, g, h to be the differenees of four quantities $\alpha, \beta, \gamma, \delta$, viz. $a, b, c, f, g, h = \beta-\gamma, \gamma-\alpha, \alpha-\beta, \alpha-\delta, \beta-\delta, \gamma-\delta$, then

$$\begin{aligned}
\cdot\quad h-g+a &= 0,\\
-h\quad \cdot+f+b &= 0,\\
g-f\ \cdot\ +c &= 0,\\
-a-b-c\ \cdot\ &= 0,
\end{aligned}$$

and taking the resolution of $(x+y)^9-x^9-y^9$, and proceeding, as on p. 46, we have

$$\begin{aligned}
\cdot\quad h^9-g^9+a^9 &= 3agh\{3(-ga+ah-hg)^3-a^2g^2h^2\},\\
-h^9\quad \cdot+f^9+b^9 &= 3bhf\{3(-hb+bf-fh)^3-b^2h^2f^2\},\\
g^9-f^9\quad \cdot+c^9 &= 3cfg\{3(-fc+cg-gf)^3-c^2f^2g^2\},\\
-a^9-b^9-c^9\quad &= 3abc\{3(bc+ca+ab)^3-a^2b^2c^2\};
\end{aligned}$$

whence

$$\begin{aligned}
&3agh(-ga+ah-hg)^3+3bhf(-hb+bf-fh)^3+3cfg(-fc+cg-gf)^3\\
&\quad+3abc(bc+ca+ab)^3-a^3g^3h^3-b^3h^3f^3-c^3f^3g^3-a^3b^3c^3=0.
\end{aligned}$$

Treating the other equations in § 1 in a similar manner, and writing

$$A = -ga + ah - hg,$$
$$B = -hb + bf - fh,$$
$$C = -fc + cg - gf,$$
$$P = \quad bc + ca + ab,$$

we have the series of identities,

$$agh \quad + bhf \quad + cfg \quad + abc \quad = 0 \ldots (2),$$
$$agh\ A + bhf\ B + cfg\ C + abc\ P = 0 \ldots (3),$$
$$agh\ A^2 + bhf\ B^2 + cfg\ C^2 + abc\ P^2 = 0 \ldots (4),$$
$$3agh\ A^3 + 3bhf\ B^3 + 3cfg\ C^3 + 3abc\ P^3$$
$$- a^3g^3h^3 - b^3h^3f^3 - c^3f^3g^3 - a^3b^3c^3 = 0 \ldots (5),$$
$$agh\ A^4 + bhf\ B^4 + cfg\ C^4 + abc\ P^4$$
$$- a^3g^3h^3A - b^3h^3f^3B - c^3f^3g^3C - a^3b^3c^3P = 0 \ldots (6),$$
$$agh\ A^5 + bhf\ B^5 + cfg\ C^5 + abc\ P^5$$
$$- 2a^3g^3h^3A^2 - 2b^3h^3f^3B^2 - 2c^3f^3g^3C^2 - 2a^3b^3c^3P^2 = 0 \ldots (7).$$

Since

$$bc + ca + ab = -\tfrac{1}{2}(a^2 + b^2 + c^2)$$
$$-ga + ah - hg = -\tfrac{1}{2}(a^2 + g^2 + h^2), \text{ \&c.,}$$

if we put

$$L = a^2 + g^2 + h^2,$$
$$M = b^2 + h^2 + f^2,$$
$$N = c^2 + f^2 + g^2,$$
$$X = a^2 + b^2 + c^2,$$

these identities may be written

$$agh \quad + bhf \quad + cfg \quad + abc \quad = 0 \ldots . (8),$$
$$agh\ L + bhf\ M + cfg\ N + abc\ X = 0 \ldots . (9),$$
$$agh\ L^2 + bhf\ M^2 + cfg\ N^2 + abc\ X^2 = 0 \ldots (10),$$
$$3agh\ L^3 + 3bhf\ M^3 + 3cfg\ N^3 + 3abc\ X^3$$
$$+ 8a^3g^3h^3 + 8b^3h^3f^3 + 8c^3f^3g^3 + 8a^3b^3c^3 = 0 \ldots (11),$$
$$agh\ L^4 + bhf\ M^4 + cfg\ N^4 + abc\ X^4$$
$$+ 8a^3g^3h^3L + 8b^3h^3f^3M + 8c^3f^3g^3N + 8a^3b^3c^3X = 0 \ldots (12),$$
$$agh\ L^5 + bhf\ M^5 + cfg\ N^5 + abc\ X^5$$
$$+ 16a^3g^3h^3L^2 + 16b^3h^3f^3M^2 + 16c^3f^3g^3N^2 + 16a^3b^3c^3X^2 = 0 \ldots (13),$$

of these (4) and (10) are Prof. Cayley's identities (p. 46).

I have not worked out any resolutions besides those given in § 1; but it would be interesting to obtain the resolutions for some values of n greater than $n = 13$, and if possible to obtain the law of formation of the residual factor; *i.e.* the remaining factor when the factors $xy(x+y)$, and x^2+xy+y^2, or $(x^2+xy+y^2)^2$, have been thrown out.

§ 4. I here add, though not connected with the subject of the paper, the identity

$$(\beta-\gamma)(\beta+\gamma-2\alpha)^3+(\gamma-\alpha)(\gamma+\alpha-2\beta)^3+(\alpha-\beta)(\alpha+\beta-2\gamma)^3=0,$$

or, if $a=\beta-\gamma$, $b=\gamma-\alpha$, $c=\alpha-\beta$ so that $a+b+c=0$,

$$a(b-c)^3+b(c-a)^3+c(a-b)^3=0.$$

A THEOREM RELATING TO PFAFFIANS.

By Prof. *H. W. Lloyd Tanner, M.A.*

THE functions to which Prof. Cayley has given the name "pfaffian" are of fundamental importance in the theory of the transformation of the expression

$$\Sigma_{i=1}^{i=n}\phi_i(x_1,\ldots x_n)\,dx_i.$$

In working at this theory I have been led to a theorem concerning pfaffians which seems of sufficient interest for publication, if, indeed, it is new.

Definitions, Notation, &c. "The definition of a pfaffian is as follows: taking the symbol 12, 13, &c. to be such that $12=-21$, &c., then we have the pfaffians (1234), (123456), &c., viz.,

$$(1234) = 12.34+13.42+14.23 \text{ (3 terms)},$$

$$\begin{aligned}(123456) = {} & 12\,(3456), \text{ where } (3456)=34.56+35.64+36.45,\\ & +13\,(4562) \qquad\qquad \&c.=\&c.,\\ & +14\,(5623)\\ & +15\,(6234)\\ & +16\,(2345),\end{aligned}$$

15 terms, each with the sign +; and so on."*

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* I have to thank Prof. Cayley for this definition.

An obvious induction shows that the transposition of a single pair of the symbols 1, 2... changes the sign of the pfaffian; whence, as in the case of determinants, it follows that a pfaffian with two identical symbols vanishes. Moreover, we may change the order of the symbols 1, 2, &c., in the above expansions; if we prefix the sign + or − to a term according as the order of the symbols 1, 2... in that term is reducible to the order 12... by an even or an odd number of transpositions.

By the symbols ({12}), ({13}),...({1234}),... will be indicated the coefficients of (12), (13),...(1234)... in the pfaffian (12...2r) which may itself be denoted by {1}. The symbol ({12}) represents a pfaffian which is derived from (12...2r) by omitting 1, 2, and so of the others; ({12...2r}) is unity. In virtue of the above we may write

$$(12...2r)\,(\{12...2r\}) = \{1\}.1$$
$$= (12).(\{12\}) + (13)(\{13\}) + ... + (1,2r)\,(\{1,2r\})$$
$$= \&c.$$

In view of these relations, and of others which will be discussed in the sequel, we may call such pairs of symbols as (12), ({12}); (1234), ({1234}) complementary; (12...2r) being the principal pfaffian.

It is easy to verify that the transposition of a pair of the numbers in ({12}), ({1234}), &c., changes the sign.

The theorem to be proved is this: Any relation between pfaffians is true of their complementaries, provided the relations be made homogeneous (*i.e.* having the same number of pfaffians in each term); and this condition can always be satisfied by introducing {12...2r} in the original, or {1} in the transformed relations where required.

Thus having

$$(12).(34) + (13)\,(42) + (14)\,(23) = (1234).1 = (1234).(\{12...2r\}),$$

there is another relation, or set of relations,

$$(\{12\})\,(\{34\}) + (\{13\}).(\{42\}) + (\{14\})\,(\{23\})$$
$$= (\{1234\}).\{1\} = (\{1234\})\,(12...2r).$$

Again, we should have

$$\{1\}\,(\{123456\}) = (\{12\})\,(\{3456\}) + (\{13\})\,(\{4562\}) + (\{14\})\,(\{5623\})$$
$$+ (\{15\})\,(\{6234\}) + (\{16\})\,(\{2345\}).$$

Consider, first, the equation

$$(\{12\}).(\{34\})+(\{13\}).(\{42\})+(\{14\}).(\{23\})-(\{1234\}).\{1\}=0\ldots(1).$$

If this be expanded in terms of (12), (13)...(23)...&c., each term will contain a factor (12), (13)...or $(1, 2r)$ involving the symbol 1. If, then, it be shown that the coefficients of (12), (13)...$(1, 2r)$ all vanish the truth of (1) will be established.

Take then the coefficient of (12). In the first term $(\{12\}).(\{34\})$ it is only in the second factor that (12) can occur [for $(\{12\})$ is a pfaffian formed from $(12\ldots2r)$ by omitting 1, 2]. In this term the coefficient of (12) is $(\{12\}).(\{1234\})$. In the second and third terms of (1), (12) does not occur, for in each case 1 is left out of the first factor and 2 out of the second. In the last term the coefficient is $-(\{1234\})(\{12\})$, which annuls the portion derived from the first term. Hence in (1) the coefficient of (12) is zero. In the same manner it may be shown that the coefficients of (13), (14) also vanish.

It is otherwise with the coefficients of (15), (16), &c.

That of (15), for example, is

$$(\{12\})(\{1534\})+(\{13\})(\{1542\})+(\{14\})(\{1523\})-(\{1234\})(\{15\}),$$

or

$$(\{12\})(\{1345\})-(\{13\})(\{1245\})+(\{14\})(\{1235\})$$
$$-(\{15\})(\{1234\})\ldots(2),$$

which does not obviously vanish. Herein the symbol 1 does not enter, but every term of the expansion will contain 2. The coefficients of (23), (24), (25) identically vanish.

If, then, the coefficients of (26), (27), &c., also vanish, (2) will vanish; in other words, the coefficient of (15) in (1) will vanish, and, since (16), (17), &c., may be dealt with in just the same way as (15), the truth of (1) will be proved. Now the coefficient of (26), say, in (2), is after re-arranging

$$(\{1234\})(\{1256\})-(\{1235\})(\{1246\})+(\{1236\})(1245\})$$
$$-(\{12\}(\{123456\}).$$

But $(\{1234\})$ is the conjugate of (34) with respect to $(\{12\})$ or $(34\ldots2r)$, and if we take this as a new principal pfaffian, the identity we have to prove may be written

$$(\{34\})(\{56\})-(\{35\})(\{46\})+(\{36\})(\{45\})-\{1\}.(\{3456\})=0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

This relation (3) differs from (1) only in that the new principal pfaffian $\{1\}$ contains two symbols less than the original $\{1\}$. But (1) is true if (3) be true; that is, (1) is true when the principal pfaffian involves $2r$ numbers, if it be true when there are only $2(r-1)$ numbers. Now (1) is true when $2r=4$, for it then becomes

$$(34).(12)+(42)(13)+(23)(14)-(1234)=0;$$

hence it is true when $2r=6$, and so generally.

In precisely the same manner the second identity may be proved. This may be written

$$(\{12\})(\{3456\})-(\{13\})(\{2456\})+(\{14\})(\{2356\})-(\{15\})(\{2346\})$$
$$+(\{16\})(\{2345\})-\{1\}.(\{123456\})=0.....(4).$$

The coefficients of (12), (13), (14), (15), (16) all obviously vanish; the coefficients of (17), (18), &c., do not so. But, taking as example (17), the coefficients of (17)(23), (17)(24), (17)(25), (17)(26), vanish; and it only remains to prove that the same is true of the coefficients of (17)(28), &c. Now the coefficient of (17)(28) is

$$-(\{12\})(\{12345678\})+(\{1238\})(\{124567\})-(\{1248\})(\{123467\})$$
$$+(\{1258\})(\{123467\})-(\{1268\})(\{123457\})+(\{1278\})(\{123456\}),$$

which if we take $(\{12\})$ as a new principal pfaffian differs from (4) only by the new $\{1\}$ containing two numbers less than the original $\{1\}$. Hence (4), being true by definition when $2r=6$, may be proved by induction to be generally true.

The results obtained amount to this; that the symbols $(\{12\}):\{1\}$, $(\{13\}):\{1\}$,...$(\{1234\}):\{1\}$, &c., satisfy the relations by which (12), (13),...(1234), &c., were defined. Hence, any theorems true of the latter are also true of the former symbols. And this is the theorem to be proved.

July, 1878.

Concerning the above paper Prof. Cayley writes:—

"I think the true form is not exactly as you put it: 'An identity subsists when the pfaffians are replaced by complementary pfaffians,' but 'If the elements of the principal pfaffian are replaced by their complementaries, then each minor pfaffian is converted into the complementary minor pfaffian'; your form being, in fact, an immediate corollary of this.

"Thus taking the principal pfaffian to be 123456, the complementaries of 12, 13, &c., are 3456, 4562, &c., and the theorem is, that substituting in any sub-pfaffian, such as 1234, the foregoing values in place of 12, 13, &c., respectively, this is changed into the complementary sub-pfaffian 56.

"Of course an identity, $1234 = 12.34 + \&c.$, subsists when the elements are changed into any other elements, viz. we have $1'2'3'4' = 1'2'.3'4' + \&c.$, and the theorem is, that writing herein $1'2' = 3456$, &c., it becomes

$$56 = 3456.1256 + \&c....$$

(where, on the left-hand, there is a factor 123456 to be inserted).

"The like theorem holds, as regards determinants, viz. for a determinant (11, 22, 33, 44, 55, 66), if each element 11, 12, &c., is replaced by the complementary first minor (22, 33, 44, 55, 66), (21, 33, 44, 55, 66), &c., then every other minor (12, 33, 54), &c., is converted into the complementary minor (21, 45, 66), &c., and the theorem for pfaffians follows at once (though the working-out would, perhaps, be rather difficult) from that for determinants, by taking a symmetrical skew determinant which is the square of a pfaffian."

NEW FORMULÆ FOR THE INTEGRATION OF

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0.$$

By Professor *Cayley*.

I HAVE found in regard to the differential equation

$$\frac{dx}{\sqrt{(a-x.b-x.c-x.d-x)}} + \frac{dy}{\sqrt{(a-y.b-y.c-y.d-y)}} = 0,$$

a system of formulæ analogous to those given, p. 63, of my *Treatise on Elliptic Functions*, for the values of $\mathrm{sn}(u+v)$, $\mathrm{cn}(u+v)$, $\mathrm{dn}(u+v)$; viz. writing for shortness

$$\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d} = a-x, b-x, c-x, d-x,$$

$$\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1, \mathrm{d}_1 = a-y, b-y, c-y, d-y,$$

and (bc, ad) to denote the determinant

$$\begin{vmatrix} 1, & x+y, & xy \\ 1, & b+c, & bc \\ 1, & a+d, & ad \end{vmatrix},$$

and (cd, ab), (bd, ac) to denote the like determinants. Then the formulæ are

$$\sqrt{\left(\frac{a-z}{d-z}\right)}=\frac{\sqrt{(a-b.a-c)}\,\{\sqrt{(\mathrm{adb_1c_1})}+\sqrt{(\mathrm{a_1d_1bc})}\}}{(bc,\,ad)},$$

$$=\frac{\sqrt{(a-b.a-c)}\,(x-y)}{\sqrt{(\mathrm{adb_1c_1})}-\sqrt{(\mathrm{a_1d_1bc})}},$$

$$=\frac{\sqrt{(a-b.a-c)}\,\{\sqrt{(\mathrm{abc_1d_1})}+\sqrt{(\mathrm{a_1b_1cd})}\}}{(a-c)\sqrt{(\mathrm{bdb_1d_1})}-(b-d)\sqrt{(\mathrm{aca_1c_1})}},$$

$$=\frac{\sqrt{(a-b.a-c)}\,\{\sqrt{(\mathrm{acb_1d_1})}+\sqrt{(\mathrm{a_1c_1bd})}\}}{(a-b)\sqrt{(\mathrm{cdc_1d_1})}-(c-d)\sqrt{(\mathrm{aba_1b_1})}},$$

$$\sqrt{\left(\frac{b-z}{d-z}\right)}=\frac{\sqrt{\left(\frac{a-b}{a-d}\right)}\,\{(a-c)\sqrt{(\mathrm{bdb_1d_1})}+(b-d)\sqrt{(\mathrm{aca_1c_1})}\}}{(bc,\,ad)},$$

$$=\frac{\sqrt{\left(\frac{a-b}{a-d}\right)}\,\{\sqrt{(\mathrm{abc_1d_1})}-\sqrt{(\mathrm{a_1b_1cd})}\}}{\sqrt{(\mathrm{adb_1c_1})}-\sqrt{(\mathrm{a_1d_1bc})}},$$

$$=\frac{\sqrt{\left(\frac{a-b}{a-d}\right)}\,(cd,\,ab)}{(a-c)\sqrt{(\mathrm{bdb_1d_1})}-(b-d)\sqrt{(\mathrm{aca_1c_1})}},$$

$$=\frac{\sqrt{\left(\frac{a-b}{a-d}\right)}\,\{(a-d)\sqrt{(\mathrm{bcb_1c_1})}+(b-c)\sqrt{(\mathrm{ada_1d_1})}\}}{(a-b)\sqrt{(\mathrm{cdc_1d_1})}-(c-d)\sqrt{(\mathrm{aba_1b_1})}},$$

$$\sqrt{\left(\frac{c-z}{d-z}\right)}=\frac{\sqrt{\left(\frac{a-c}{a-d}\right)}\,\{(a-b)\sqrt{(\mathrm{cdc_1d_1})}+(c-d)\sqrt{(\mathrm{aba_1b_1})}\}}{(bc,\,ad)},$$

$$=\frac{\sqrt{\left(\frac{a-c}{a-d}\right)}\,\{\sqrt{(\mathrm{acb_1d_1})}-\sqrt{(\mathrm{a_1c_1bd})}\}}{\sqrt{(\mathrm{adb_1c_1})}-\sqrt{(\mathrm{a_1d_1bc})}},$$

$$=\frac{\sqrt{\left(\frac{a-c}{a-d}\right)}\,\{(a-d)\sqrt{(\mathrm{bcb_1c_1})}-(b-c)\sqrt{(\mathrm{ada_1d_1})}\}}{(a-c)\sqrt{(\mathrm{bdb_1d_1})}-(b-d)\sqrt{(\mathrm{aca_1c_1})}},$$

$$=\frac{\sqrt{\left(\frac{a-c}{a-d}\right)}\,(bd,\,ac)}{(a-b)\sqrt{(\mathrm{cdc_1d_1})}-(c-d)\sqrt{(\mathrm{aba_1b_1})}},$$

The twelve equations are equivalent to each other, each giving z as one and the same function of x, y; and regarding z as a constant of integration, any one of the equations is a form of the integral of the proposed differential equation.

Writing in the formulæ $x = a, b, c, d$ successively, the formulæ become

$x = a$	$x = b$	$x = c$	$x = d$
$\frac{a-z}{d-z} = \frac{a_1}{d_1}$,	$-\frac{c-a}{d-b}\frac{b_1}{c_1}$,	$-\frac{b-a}{d-c}\frac{c_1}{b_1}$,	$\frac{a-b.a-c}{d-b.d-c}\frac{d_1}{a_1}$,
$\frac{b-z}{d-z} = \frac{b_1}{d_1}$,	$-\frac{c-b}{d-a}\frac{a_1}{c_1}$,	$\frac{b-a.b-c}{d-a.d-c}\frac{d_1}{b_1}$,	$-\frac{a-b}{d-c}\frac{c_1}{a_1}$,
$\frac{c-z}{d-z} = \frac{c_1}{d_1}$,	$\frac{c-a.c-b}{d-a.d-b}\frac{d_1}{c_1}$,	$-\frac{b-c}{d-a}\frac{a_1}{b_1}$,	$-\frac{a-c}{d-b}\frac{b_1}{a_1}$,

viz. in the first case we have $z = y$, and in each of the other cases z equal to a linear function $\frac{\alpha y + \beta}{\gamma y + \delta}$ of y.

Cambridge, July 3, 1878.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, June, 13th, 1878.—Prof. H. J. S. Smith, F.R.S., *Vice-President*, in the chair. Mr. T. R. Terry was admitted into the Society and Mr. J. D. H. Dickson, M.A., Fellow and Tutor of St. Peter's College, Cambridge, was proposed for election. Dr. Hirst, F.R.S., communicated a paper by M. Halphen "On the characteristics of systems of Conics." Mr. J. J. Walker read a paper "On a method in the Analysis of Plane Curves." He discussed, by means of the method, the problem of the Inflexion-tangential curve for the quartic. Mr. Tucker communicated the following papers: "On the Calculus of Equivalent statements," Mr. Hugh McColl; "On the Flexure of Spaces," Mr. C. J. Monro; "On the Decomposition of certain numbers into sums of two square Integers by Continued Fractions," Mr. S. Roberts, F.R.S.; "On a new method of finding Differential Resolvents of Algebraical Equations," Mr. R. Rawson. Questions were asked by Prof. Cayley, F.R.S., ("Has a proof been given of the statement that in colouring the map of a country, divided into counties, only four distinct colours are required, so that no two adjacent counties should be painted in the same colour?"); by Mr. Merrifield, F.R.S., on the uniform distribution of points in space; by Mr. Tucker, with reference to exceptions to Fermat's statement that $2^{2^m} + 1$ is a prime (see *Nature*, No. 447, for the recent discovery that in addition to the well-known exception of $2^{2^5} + 1$, $2^{2^{12}} + 1$ is also an exception).

R. TUCKER, M.A., *Hon. Sec.*

NOTE ON CERTAIN THEOREMS IN DEFINITE INTEGRATION.

By *J. W. L. Glaisher.*

§ 1. THIS note refers to Boole's Memoir "On the comparison of Transcendents, with certain applications to the theory of definite integrals" (*Phil. Trans.*, vol. CXLVII., pp. 745–803, 1857), and two papers of my own "On a theorem in definite integration," *Quarterly Journal*, vol. X., pp. 347–356, 1870, and "On the reduction of functional transcendents" (*Messenger*, vol I., pp. 153–162, 1872); it contains transformations and examples relating to the contents of these papers.

§ 2. The theorem*

$$\int_{-\infty}^{\infty} f\left(x - \frac{a}{x+\lambda}\right) dx = \int_{-\infty}^{\infty} f(x)\, dx,$$

(λ real, a real and positive) may be put in a slightly different form; for c being real, but otherwise unrestricted, the left-hand side may be written

$$\int_{-\infty}^{\infty} f\left(x + c - \frac{a}{x+\lambda}\right) dx;$$

Now
$$x - \frac{a}{\lambda} - \frac{a}{x+\lambda} = \frac{x^2 + \left(\lambda + \frac{a}{\lambda}\right) x}{x+\lambda},$$

so that
$$\int_{-\infty}^{\infty} f\left(x \frac{x+n\lambda}{x+\lambda}\right) dx = \int_{-\infty}^{\infty} f(x)\, dx,$$

(λ real, n real, positive, and not less than unity).

Boole's general theorem when transformed as above, is that

$$\int_{-\infty}^{\infty} f\left\{x\left(A_1 \frac{x+n_1\lambda_1}{x+\lambda_1} + A_2 \frac{x+n_2\lambda_2}{x+\lambda_2} \ldots + A_r \frac{x+n_r\lambda_r}{x+\lambda_r}\right)\right\} dx$$
$$= \frac{1}{A_1 + A_2 \ldots + A_r} \int_{-\infty}^{\infty} f(x)\, dx,$$

* *Phil. Trans.*, vol. CXLVII. p. 780.

($\lambda_1, \lambda_2 \ldots \lambda_r$ real; $n_1, n_2 \ldots n_r$ real positive, and not less than unity; $A_1, A_2, \ldots A_r$ all of the same sign). Whether $A_1, A_2, \ldots A_r$ are positive or negative they are to be taken as positive on the right-hand side of the equation, *i.e.* the factor is really $\sqrt{(A_1 + A_2 \ldots + A_r)^{-2}}$.

§ 3. We know that

$$\tan x = -\frac{2x}{x^2-(\frac{1}{2}\pi)^2} - \frac{2x}{x^2-(\frac{3}{2}\pi)^2} - \frac{2x}{x^2-(\frac{5}{2}\pi)^2} - \&c. \ldots\ldots (1),$$

and

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \&c.,$$

whence

$$1 - \frac{\tan x}{x} = 2\left\{\frac{1}{(\frac{1}{2}\pi)^2} + \frac{1}{(\frac{3}{2}\pi)^2} + \frac{1}{(\frac{5}{2}\pi)^2} + \&c. \right.$$
$$\left. + \frac{1}{x^2-(\frac{1}{2}\pi)^2} + \frac{1}{x^2-(\frac{3}{2}\pi)^2} + \frac{1}{x^2-(\frac{5}{2}\pi)^2} + \&c.\right\}$$
$$= \frac{2x^2}{(\frac{1}{2}\pi)^2\{x^2-(\frac{1}{2}\pi)^2\}} + \frac{2x^2}{(\frac{3}{2}\pi)^2\{x^2-(\frac{3}{2}\pi)^2\}} + \frac{2x^2}{(\frac{5}{2}\pi)^2\{x^2-(\frac{5}{2}\pi)^2\}} + \&c.$$
$$\ldots\ldots\ldots\ldots\ldots\ldots (12),$$

and

$$\frac{1}{x}\left(1 - \frac{\tan x}{x}\right) = \frac{1}{(\frac{1}{2}\pi)^2}\left(\frac{1}{x-\frac{1}{2}\pi} + \frac{1}{x+\frac{1}{2}\pi}\right)$$
$$+ \frac{1}{(\frac{3}{2}\pi)^2}\left(\frac{1}{x-\frac{3}{2}\pi} + \frac{1}{x+\frac{3}{2}\pi}\right) + \&c. \ldots (3),$$

when by the theorem (*Quarterly Journal*, vol. x. p. 349),

$$\int_{-\infty}^{\infty} f\left(\frac{x-\tan x}{x^2}\right) dx = \left\{\frac{2}{(\frac{1}{2}\pi)^2} + \frac{2}{(\frac{3}{2}\pi)^2} + \&c.\right\} \int_{-\infty}^{\infty} \frac{f(v)}{v^2}\, dv$$
$$= \int_{-\infty}^{\infty} \frac{f(v)}{v^2}\, dv.$$

This example is similar to, but somewhat simpler than, that contained in Example I (*Quarterly Journal*, vol. x. p. 350).

Boole's theorem gives that

$$\int_{-\infty}^{\infty} f\left(\frac{x^3 - x + \tan x}{x^2}\right) dx = \int_{-\infty}^{\infty} f(v)\, dv.$$

We have from (2)

$$\frac{1}{x}\left(1-\frac{\tan\sqrt{x}}{\sqrt{x}}\right)=\frac{2}{(\frac{1}{2}\pi)^2\{x-(\frac{1}{2}\pi)^2\}}+\frac{2}{(\frac{3}{2}\pi)^2\{x-(\frac{3}{2}\pi)^2\}}+\&c.$$

whence

$$\int_{-\infty}f\left\{\frac{1}{x}\left(1-\frac{\tan\sqrt{x}}{\sqrt{x}}\right)\right\}dx=\int_{-\infty}^{\infty}\frac{f(v)}{v^2}dv\ldots\ldots\ldots(4),$$

and we may notice that from the original series (1), Boole's theorem gives that

$$\int_{-\infty}^{\infty}f\left(x+\frac{\tan\sqrt{x}}{\sqrt{x}}\right)dx=\int_{-\infty}^{\infty}f(v)\,dv\ldots\ldots\ldots\ldots(5).$$

§4. From (1)

$$\tan\frac{1}{x}=\frac{2x}{(\frac{1}{2}\pi)^2x^2-1}+\frac{2x}{(\frac{3}{2}\pi)^2x^2-1}+\frac{2x}{(\frac{5}{2}\pi)^2x^2-1}+\&c.$$

whence

$$\frac{1}{\sqrt{x}}\tan\frac{1}{\sqrt{x}}=\frac{1}{(\frac{1}{2}\pi)^2}\frac{2}{x-(\frac{1}{2}\pi)^{-2}}+\frac{1}{(\frac{3}{2}\pi)^2}\frac{2}{x-(\frac{3}{2}\pi)^{-2}}+\&c.,$$

and, therefore,

$$\int_{-\infty}^{\infty}f\left(\frac{1}{\sqrt{x}}\tan\frac{1}{\sqrt{x}}\right)dx=\int_{-\infty}^{\infty}\frac{f(v)}{v^2}\,dv\ldots\ldots\ldots(6),$$

viz.
$$\int_{-\infty}^{\infty}f(\sqrt{x}\tan\sqrt{x})\,\frac{dx}{x^2}=\int_{-\infty}^{\infty}\frac{f(v)}{v^2}\,dv\ldots\ldots\ldots(7),$$

§5. Boole's theorem, applied to the equation

$$\frac{\cot x}{x}-\frac{1}{x^2}+\tfrac{1}{3}=\frac{1}{\pi^2}\frac{2x^2}{x^2-\pi^2}+\frac{1}{(2\pi)^2}\frac{2x^2}{x^2-(2\pi)^2}+\&c.,$$

gives
$$\int_{-\infty}^{\infty}f\left(x+\frac{1}{x}-\frac{\cot\sqrt{x}}{\sqrt{x}}\right)dx=\int_{-\infty}^{\infty}f(v)\,dv\ldots\ldots\ldots(8),$$

and from the *Quarterly Journal* theorem, we have

$$\int_{-\infty}^{\infty}f\left\{\frac{1}{x}\left(\frac{\cot\sqrt{x}}{\sqrt{x}}-\frac{1}{x}+\tfrac{1}{3}\right)\right\}dx=\tfrac{1}{3}\int_{-\infty}^{\infty}\frac{f(v)}{v^2}\,dv\ldots(9).$$

§6. It is to be remarked, as in *Messenger*, vol. I. p. 158, that when the expression under the integral sign changes its form within the limits of integration {as *ex. gr.* in

(4), (5), ...(9)}, the equation is really a theorem in the comparison of transcendents; thus *ex. gr.* (7) is that

$$\int_0^\infty f(\sqrt{x}\tan\sqrt{x})\frac{dx}{x^2}+\int_{-\infty}^0 f(\sqrt{x}\tan\sqrt{x})\frac{dx}{x^2}=\int_{-\infty}^\infty \frac{f(v)}{v^2}\,dv,$$

and the second integral

$$=\int_0^\infty f(i\sqrt{x}\tan i\sqrt{x})\frac{dx}{x^2},$$

so that the theorem is

$$\int_0^\infty f(\sqrt{x}\tan\sqrt{x})\frac{dx}{x^2}+\int_0^\infty f(-\sqrt{x}\tanh\sqrt{x})\frac{dx}{x^2}=\int_{-\infty}^\infty \frac{f(v)}{v^2}\,dv,$$

For example, taking $f(v)=v^2e^{-v^2}$

$$\int_0^\infty \tan^2\sqrt{x}\exp(-x\tan^2\sqrt{x})\frac{dx}{x}$$
$$+\int_0^\infty \tanh^2\sqrt{x}\exp(-x\tanh^2\sqrt{x})\frac{dx}{x}=\sqrt{\pi},$$

where $\exp u$ is printed for e^u.

§ 7. With regard to Boole's theorem and that in vol. x. of the *Quarterly Journal*, there is one point that deserves notice. Boole obtains his theorem by means of the transforming equation

$$x-\frac{a_1}{x-\lambda_1}-\frac{a_2}{x-\lambda_2}\ldots-\frac{a_n}{x-\lambda_n}=v,$$

and, supposing $\lambda_1, \lambda_2, \ldots\lambda_n$ to be in ascending order of magnitude, as v passes from $-\infty$ to ∞ the least root passes from $-\infty$ to λ_1, the next from λ_1 to λ_2..., and the greatest from λ_n to ∞. Now when we take the transforming equation to be

$$x-\frac{1}{x-\pi}-\frac{1}{x+\pi}-\frac{1}{x-2\pi}-\frac{1}{x+2\pi}\ldots-\frac{1}{x-n\pi}-\frac{1}{x+n\pi}=v,$$

the least root passes from $-\infty$ to $-n\pi$, the next from $-n\pi$ to $-(n-1)\pi$ and so on. In the limit we take n infinite, and since the least root has to pass from $-\infty$ (the limit of the integral) to $-n\pi$ (n infinite), it is clear that we, in effect, assume the infinity of the limit of the integral to exceed the (numerically) greatest multiple of π for which $\cot x$ is infinite. This is perfectly allowable whenever the function f is such that $f(x)$ becomes zero when x is infinite, for the

quantity under the integral sign vanishing near the limits of the integral, the numerical value of the integral cannot be effected by an assumption of this kind with regard to the infinities of the limits.

The *Quarterly Journal* theorem is obtained by means of the transforming equation

$$\frac{a_1}{x-\lambda_1}+\frac{a_2}{x-\lambda_2}\ldots+\frac{a_n}{x-\lambda_n}=v,$$

and the greatest root passes from λ_n to ∞, changes sign and passes from $-\infty$ to λ_1 as v changes from ∞ to $-\infty$.

Thus, on the whole, it appears that in any one of the equations, when both the integrals are finite in value, and independent of the particular infinities which form the ∞'s of the limits, *i.e.* such that the values of the integrals are not altered by changing the ∞ of any of the limits into $\infty+h$, or ∞^2 &c., then the equation will certainly be true; but that when these conditions are not satisfied, a special examination of the case is requisite. If the integrals become infinite for particular values of x or v their principal values are to be taken.

It is to be observed that all the results in §§ 1-6 are derived from the theorems

$$\int_{-\infty}^{\infty} f(x-\chi x)\,dx=\int_{-\infty}^{\infty} f(v)\,dv,$$

and

$$\int_{-\infty}^{\infty} f(\chi x)\,dx=\int_{-\infty}^{\infty}\frac{f(v)}{v^2}\,dv,$$

where, for brevity,

$$\frac{a_1}{x-\lambda_1}+\frac{a_2}{x-\lambda_2}\ldots+\frac{a_n}{x-\lambda_n}$$

is denoted by χx; and that these can be proved directly without the use of Boole's symbol Θ, as in Prof. Cayley's note, *Quarterly Journal*, vol. x. p. 356. The other theorems in Boole's paper and in the *Quarterly Journal*, in which the integrals are of the form

$$\int_{-\infty}^{\infty}\phi(x)f(x-\chi x)\,dx \text{ and } \int_{-\infty}^{\infty}\phi(x)f(\chi x)\,dx$$

were obtained by means of Θ, one of the operations of which is to subtract the coefficients of x^{-1} in the expansion of a certain expression in descending powers of x. When χx

contains an infinite number of terms, the expansion in descending powers of x is not always possible, and before replacing χx by $\cot x$, or $\frac{1}{\sqrt{x}} \tan \frac{1}{\sqrt{x}}$, or other such functions, in these theorems, it is necessary to examine each case separately, and determine whether the theorem is applicable. For an example in which the substitution is not permissible, see *Messenger*, vol. I. p. 153 (1872). I do not here consider any of the equations, similar to those obtained in §§ 1-6, that can be obtained by these theorems.

§ 8. Boole has shown (p. 787 of his memoir) that

$$\int_{-\infty}^{\infty} \frac{f(x-\chi x)}{x-h}\,dx = \int_{-\infty}^{\infty} \frac{f(v)}{v-\chi h}\,dv,$$

where χx has the same meaning as in the last section, and by means of this theorem, by putting $h = bi$, and $h = -bi$, and subtracting and adding the results, we obtain reductions for the integrals

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+b^2} f(x-\chi x) \text{ and } \int_{-\infty}^{\infty} \frac{x\,dx}{x^2+b^2} f(x-\chi x).$$

Taking the simplest case of $\chi x = a^2 : x$, we find

$$\int_{-\infty}^{\infty} \frac{dx}{x-bi} f\left(x-\frac{a^2}{x}\right) = \int_{-\infty}^{\infty} \frac{f(v)}{v-\left(b+\frac{a^2}{b}\right)i}\,dv \ldots (10),$$

whence

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+b^2} f\left(x-\frac{a^2}{x}\right) = \left(1+\frac{a^2}{b^2}\right) \int_{-\infty}^{\infty} \frac{f(v)\,dv}{v^2+\left(b+\frac{a^2}{b}\right)^2} \ldots (11),$$

$$\int_{-\infty}^{\infty} \frac{x\,dx}{x^2+b^2} f\left(x-\frac{a^2}{x}\right) = \int_{-\infty}^{\infty} \frac{v f(v)\,dv}{v^2+\left(b+\frac{a^2}{b}\right)^2} \ldots\ldots (12).$$

These formulæ can also be deduced from

$$\int_{-\infty}^{\infty} \frac{dx}{x-h} f(\chi x) = \int_{-\infty}^{\infty} \frac{\chi h\,dv}{\chi h - v} \frac{f(v)}{v},$$

(*Quarterly Journal*, vol. X. p. 351); for, taking

$$\chi(x) = \tfrac{1}{2} \frac{1}{x-a} + \tfrac{1}{2} \frac{1}{x+a} = \frac{1}{x-\frac{a^2}{x}};$$

we have

$$\int_{-\infty}^{\infty} \frac{dx}{x-bi} f\left\{\frac{1}{x^2-\frac{a^2}{x}}\right\} = \int_{-\infty}^{\infty} \frac{dv}{1-v\left(b+\frac{a^2}{b}\right)i} \frac{f(v)}{v}$$

$$= \int_{-\infty}^{\infty} \frac{dv}{v-\left(b+\frac{a^2}{b}\right)i} f\left(\frac{1}{v}\right),$$

agreeing with (10).

§9. As the remaining portion of this paper will be chiefly devoted to evaluations derived from (11), I here give another investigation of this equation derived from the formula

$$\int_{-\infty}^{\infty} x^m f\left(x-\frac{a^2}{x}\right) dx$$

$$= \Sigma_{n=0}^{n=m} \frac{(2m+1)(2m+2)\ldots(n+m)}{1.2.3\ldots(n-m)} a^{2n-m} \int_{-\infty}^{\infty} v^{2m} f(v) dv \ldots (13),$$

which is proved by Boole on p. 783 of of his memoir. If we put $a=1$ this becomes the formula given by Cauchy in his *Exercices de Mathématiques*, t. I. p. 54 (1826); and (13) can be readily deduced from Cauchy's theorem. Another proof of (13) is also given in the *Messenger*, vol. II. pp. 78, 79 (1872); it is there remarked that (13) does not follow from Cauchy's result, but this statement is corrected in vol. III. p. 9 (1876), of the *Proceedings* of the Cambridge Philosophical Society.*

Putting in (13) $n=1, 2, 3\ldots$ we have

$$\int_{-\infty}^{\infty} f(X)\,dx = \int_{-\infty}^{\infty} dv f(v) \left\{1 \qquad\qquad\qquad\qquad\right\},$$

$$\int_{-\infty}^{\infty} x^2 f(X)\,dx = \int_{-\infty}^{\infty} dv f(v) \left\{a^2 + \qquad v^2 \qquad\qquad\right\},$$

$$\int_{-\infty}^{\infty} x^4 f(X)\,dx = \int_{-\infty}^{\infty} dv f(v) \left\{a^4 + \quad 3a^2v^2 + \quad v^4 \quad\right\},$$

$$\int_{-\infty}^{\infty} x^6 f(X)\,dx = \int_{-\infty}^{\infty} dv f(v) \left\{a^6 + \frac{3.4}{1.2}\ a^4v^2 + 5a^2v^4 + v^6\right\},$$

.....

* "On a formula of Cauchy's for the Evaluation of a class of Definite Integrals," *Camb. Phil. Soc., Proc.* vol. III. pp. 5-12. The paper relates to Cauchy's own proof of (13), and to a similar theorem.

where $$X = x - \frac{a^2}{x}.$$

Multiplying by b^{-2}, $-b^{-4}$, b^{-6}...and adding

$$\int_{-\infty}^{\infty} \frac{1}{b^2}\left(1+\frac{x^2}{b^2}\right)^{-1} f(X)\,dx = \int_{-\infty}^{\infty} dv f(v)$$

$$\times \left\{\frac{1}{b^2}\left(1+\frac{a^2}{b^2}\right)^{-1} - \frac{1}{b^4}\left(1+\frac{a^2}{b^2}\right)^{-2} v^2 + \frac{1}{b^6}\left(1+\frac{a^2}{b^2}\right)^{-3} v^4 - \&c.\right\},$$

whence

$$\int_{-\infty}^{\infty} \frac{f(X)\,dx}{x^2+b^2} = \int_{-\infty}^{\infty} \frac{dv f(v)}{a^2+b^2}\left\{1 - \frac{b^2v^2}{(a^2+b^2)^2} + \frac{b^4v^4}{(a^2+b^2)^4} - \&c.\right\}$$

$$= (a^2+b^2)\int_{-\infty}^{\infty} \frac{f(v)\,dv}{b^2v^2+(a^2+b^2)^2},$$

and, therefore,

$$\int_{-\infty}^{\infty} \frac{f(X)\,dx}{x^2+b^2} = \left(1+\frac{a^2}{b^2}\right)\int_{-\infty}^{\infty} \frac{f(v)\,dv}{v^2+\left(b+\frac{a^2}{b}\right)^2},$$

agreeing with (11). Thus, if we denote by $\phi(b)$ the value of the integral

$$\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x^2+b^2},$$

then

$$\int_{-\infty}^{\infty} \frac{f\left(x-\frac{a^2}{x}\right)}{x^2+b^2}\,dx = \left(1+\frac{a^2}{b^2}\right)\phi\left(b+\frac{a^2}{b}\right).$$

§ 10. In the investigation in the last section two points may be noticed. First, although the series

$$1 - kx^2 + k^2x^4 - \&c. = (1+kx^2)^{-1},$$

only when kx^2 is less than unity, still we have

$$\int_{-\infty}^{\infty} \frac{f(x)}{1+kx^2}\,dx = \int_{-\infty}^{\infty} f(x)\,dx - k\int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$+ k^2\int_{-\infty}^{\infty} x^4 f(x)\,dx - \&c.,$$

if k is less than unity and $\int_{-\infty}^{\infty} x^n f(x)\,dx$ does not become infinite for any positive even value of n, for we may regard the right-hand side of the equation as the expansion of the integral on the left-hand side, given by Taylor's Theorem.

It has been assumed that b is greater than unity (otherwise the series on the left-hand side might be divergent), but this restriction is easily removed by transforming the integrals by putting $x=\mu x'$, $v=\mu v'$, and taking $a'=a:\mu$, $b'=b:\mu$. Secondly, the theorem has been proved for values of $f(x)$ such that $\int_{-\infty}^{\infty} x^n f(X)\,dx$, and $\int_{-\infty}^{\infty} v^n f(v)\,dx$ are never infinite, viz. *ex. gr.* we have

$$\int_{-\infty}^{\infty} \frac{e^{-kX^2}F(X)}{x^2+b^2}\,dx = \frac{B}{b}\int_{-\infty}^{\infty} \frac{e^{-kv^2}F(v)}{v^2+B^2}\,dv \quad \ldots\ldots(14),$$

where $$X=x-\frac{a^2}{x}, \quad B=b+\frac{a^2}{b}.$$

Now regarding this merely as a result, without reference to the manner in which it has been obtained, if we gradually diminish k indefinitely, the limiting equation is

$$\int_{-\infty}^{\infty} \frac{F(X)\,dx}{x^2+b^2} = \frac{B}{b}\int_{-\infty}^{\infty} \frac{F(v)\,dv}{v^2+B^2} \quad \ldots\ldots\ldots(15),$$

and, *if both these integrals are finite*, there is no reason to suppose a discontinuity takes place when $k=0$ in (14); and thus we obtain (15), or at all events (15) is true in the sense of being the limiting form of every equation, similar to (14), in which f is any function such that $\int_{-\infty}^{\infty} x^n \psi(X) F(X)\,dx$, and $\int_{-\infty}^{\infty} v^n \psi(v) F(v)\,dx$ are never infinite.

§ 11. If $f(x)$ is an even function of x, then

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\int_{-\infty}^{\infty} f(x)\,dx;$$

and in the equations we may take the limits to be ∞ and 0.

From the integrals

$$\int_0^{\infty} \frac{\cos cx}{x^2+b^2}\,dx = \frac{\pi}{2b}e^{-cb}, \quad \int_0^{\infty} \frac{x\sin cx}{x^2+b^2}\,dx = \frac{\pi}{2}e^{-cb},$$

we deduce by (11) and (12)

$$\int_0^\infty \frac{\cos c\left(x-\frac{a^2}{x}\right)}{x^2+b^2}\,dx=\frac{\pi}{2b}\exp\left\{-c\left(b+\frac{a^2}{b}\right)\right\},$$

$$\int_0^\infty \frac{x\sin c\left(x-\frac{a^2}{x}\right)}{x^2+b^2}\,dx=\frac{\pi}{2}\exp\left\{-c\left(b+\frac{a^2}{b}\right)\right\},$$

and, putting*

$$\operatorname{erfc}x=\int_x^\infty \exp(-x^2)\,dx,$$

we have

$$\int_0^\infty \frac{\exp(-c^2x^2)}{x^2+b^2}\,dx=\frac{\sqrt{\pi}}{b}\exp(c^2b^2)\operatorname{erfc}cb,$$

whence

$$\int_0^\infty \frac{\exp\left\{-c^2\left(x^2+\frac{a^4}{x^2}\right)\right\}}{x^2+b^2}\,dx$$
$$=\frac{\sqrt{\pi}}{b}\exp\left\{c^2\left(b^2+\frac{a^4}{b^2}\right)\right\}\operatorname{erfc}c\left(b+\frac{a^2}{b}\right).$$

To verify this result, put $a^2=c^{-1}$ and replace c^2 by c; the equation thus becomes

$$\int_0^\infty \frac{\exp(-cx^2-x^{-2})}{x^2+b^2}\,dx=\frac{\sqrt{\pi}}{b}\exp\left(cb^2+\frac{1}{b^2}\right)\operatorname{erfc}\left(b\sqrt{c}+\frac{1}{b}\right),$$

differentiating with regard to c, changing the signs and adding the original equation, we obtain

$$\int_0^\infty \exp\left(-cx^2-\frac{1}{x^2}\right)dx=\frac{\sqrt{\pi}}{2\sqrt{c}}e^{-2\sqrt{c}},$$

which is a well-known evaluation.

§ 12. In the *Messenger* t. i. p. 156, it is shown that

$$\int_{-\infty}^\infty \frac{f(\cot x)}{x^2+b^2}\,dx=\frac{1}{b}\coth b\int_{-\infty}^\infty \frac{f(v)\,dv}{v^2+\coth^2 b},$$

---

* In the *Messenger*, vol. I. p. 157, $\operatorname{erf}x$ is defined as $\int_x^\infty \exp(-x^2)\,dx$; but it is more convenient to take

$$\operatorname{erf}x=\int_0^x \exp(-x^2)\,dx,\quad \operatorname{erfc}x=\int_x^\infty \exp(-x^2)\,dx;$$

the two functions are connected by the equation $\operatorname{erf}x+\operatorname{erfc}x=\frac{1}{2}\sqrt{\pi}$.

whence, from (11),

$$\int_{-\infty}^{\infty} \frac{f\left\{\cot\left(x-\frac{a^2}{x}\right)\right\}}{x^2+b^2}\,dx = \left(1+\frac{a^2}{b^2}\right)\int_{-\infty}^{\infty} \frac{f(\cot v)}{v^2+\left(b+\frac{a^2}{b}\right)^2}\,dv$$

$$= \frac{1}{b}\coth\left(b+\frac{a^2}{b}\right)\int_{-\infty}^{\infty} \frac{f(v)\,dv}{v^2+\coth^2\left(b+\frac{a^2}{b}\right)} \quad \ldots(16),$$

and writing as before

$$X = x-\frac{a^2}{x}, \quad B = b+\frac{a^2}{b},$$

we have, by applying (16) to the examples on p. 157 of the paper referred to at the beginning of this section,

$$\int_0^{\infty} \frac{\cos(c\cot X)}{x^2+b^2}\,dx = \frac{\pi}{2b}\exp(-c\coth B),$$

$$\int_0^{\infty} \frac{\exp(-c^2\cot^2 X)}{x^2+b^2}\,dx = \frac{\sqrt{\pi}}{b}\exp(c^2\coth^2 B)\,\operatorname{erfc}(c\coth B),$$

$$\int_0^{\infty} \frac{\exp(-c^2\cot^2 X)\cos(2g\cot X)}{x^2+b^2}\,dx = \frac{\sqrt{\pi}}{2b}\exp(c^2\coth^2 B)$$

$$\times\left\{\exp(-2g\coth B)\operatorname{erfc}\left(c\coth B-\frac{g}{c}\right)\right.$$

$$\left.+\exp(2g\coth B)\operatorname{erfc}\left(c\coth B+\frac{g}{c}\right)\right\},$$

13. While referring to the paper in vol. I. of the *Messenger*, I may here point out two errata in it, viz., (i) on p. 154, the limits in equation (3) and in the line above should be π and 0, instead of ∞ and $-\infty$; the theorem being that

$$\int_{-\infty}^{\infty} \phi(x) f(\cot x)\,dx = \int_0^{\pi} \psi(x) f(\cot x)\,dx,$$

where

$$\psi(x) = \phi(x)+\phi(x-\pi)+\phi(x+\pi)+\&c.,$$

and (ii) in equation (12), p. 157, $\coth a$ should be substituted for a in the expressions subject to the functional sign erf.

§ 14. It is easy to deduce from (11) a similar formula on which the denominator is x^4+b^4; for replacing b^2 by b^2i, and using the formulæ

$$\frac{1}{x^2-b^2i}+\frac{1}{x^2+b^2i}=\frac{2x^2}{x^4+b^4},$$

$$\frac{1}{x^2-b^2i}-\frac{1}{x^2+b^2i}=\frac{2b^2i}{x^4+b^4},$$

we find

$$\int_{-\infty}^{\infty}\frac{f\left(x-\frac{a^2}{x}\right)}{x^4+b^4}\,dx=\int_{-\infty}^{\infty}\frac{a^2v^2+a^4+b^4}{b^4\left(v^2+2a^2\right)^2+\left(a^4-b^4\right)^2}\,dv,$$

$$\int_{-\infty}^{\infty}x^2\frac{f\left(x-\frac{a^2}{x}\right)}{x^4+b^4}\,dx=\int_{-\infty}^{\infty}\frac{b^4v^2+a^6+a^2b^4}{b^4\left(v^2+2a^2\right)^2+\left(a^4-b^4\right)^2}\,dv.$$

These results can be verified by putting $a=b$, for the second equation then becomes

$$\int_{-\infty}^{\infty}\frac{f\left(x-\frac{a^2}{x}\right)}{x^2+\frac{a^4}{x^2}}\,dx=\int_{-\infty}^{\infty}\frac{f(v)\,dv}{v^2+2a^2},$$

viz.

$$\int_{-\infty}^{\infty}\frac{f(X)}{X^2+2a^2}\,dx=\int_{-\infty}^{\infty}\frac{f(v)\,dv}{v^2+2a^2};$$

which is a case of the fundamental theorem. The first equation becomes

$$\int_{-\infty}^{\infty}\frac{f(X)}{X^2+2a^2}\,\frac{dx}{x^2}=\frac{1}{a^2}\int_{-\infty}^{\infty}\frac{f(v)\,dv}{v^2+2a^2},$$

which is true, since

$$\int_{-\infty}^{\infty}f\left(x-\frac{a^2}{x}\right)\frac{dx}{x^2}=\frac{1}{a^2}\int_{-\infty}^{\infty}f(v)\,dv.$$

Similarly from (12) we have

$$\int_{-\infty}^{\infty}x\,\frac{f\left(x-\frac{a^2}{x}\right)}{x^4+b^4}\,dx=\left(b^4-a^4\right)\int_{-\infty}^{\infty}\frac{vf(v)\,dv}{b^4\left(v^2+2a^2\right)^2+\left(a^4-b^4\right)^2}$$

$$\int_{-\infty}^{\infty}x^3\frac{f\left(x-\frac{a^2}{x}\right)}{x^4+b^4}\,dx=b^4\int_{-\infty}^{\infty}\frac{\left(v^2+2a^2\right)vf(v)\,dv}{b^4\left(v^2+2a^2\right)^2+\left(a^4-b^4\right)^2}.$$

ON A CASE OF WAVE MOTION.

By Professor *C. Niven.*

WAVE motion is essentially a case of continual transformations of kinetic and potential energy, the kinetic energy which any set of particles acquired as the wave passes over them being due to changes either in the amount of potential energy which they possess or of the kinetic and potential energy of the system with which they are connected, or it may be due to both of these causes. This view of wave motion is, in general, masked by the ordinary method of solving questions of this sort by means of differential equations and arbitrary functions. It is interesting, therefore, to find a case in which we can dispense with this method and exhibit the successive transformations which take place at each element of matter as the wave passes over it.

We can do so in the following problem, due to Mr. C. H. Prior, of Pembroke College, Cambridge: *An elastic filament is hung from one end and cut at any point; it is required to determine the subsequent motion of each part.*

The initial distribution of each being of displacements, we shall assume that any solution which satisfies the principle of conservation of energy and the initial conditions is *the* solution of the problem.

2. Let AB (fig. 5) be the filament, and let the part $BC(=l)$ be cut off; we shall first discuss the motion of AC.

If any element dx of the unstretched string be stretched under the influence of a stress T, which may be either tensile or contractile, the extension or contraction will be $\frac{T}{\lambda}dx$, where λ is Young's modulus, and the potential energy stored up in doing so will be $\frac{1}{2}\frac{T^2}{\lambda}dx$. The weight of the element will be $mgdx$; but it will simplify somewhat the symbols to assume that the unit of mass is the unit of length of the string. We may, therefore, put $m=1$, but the results obtained for the velocities of the points of the string and of the propagation of the waves may be reduced to ordinary units by writing $\frac{\lambda}{m}$ instead of λ.

Consider, first, the lowest element of the string, AC whose unstretched length is dx; before the instant of cutting it is under a tension $= gl$, while after cutting its tension is zero. The amount of energy set free is, therefore, $\frac{1}{2}\frac{g^2l^2}{\lambda}dx$, and this is converted into kinetic energy $\frac{1}{2}V^2dx$, where

$$V=\frac{gl}{\sqrt{\lambda}}.$$

The subsequent motion of the string AC will be of the following character: each element will, in succession, assume the velocity V, all those below it retaining this velocity and the distribution of tensions in the part over which the wave has passed will be the same as if the string AC were hanging freely. To verify this, consider what takes place in the element PQ (fig. 6), as the wave passes over it, the unstretched lengths of CP, PQ being x, dx.

Before the wave reaches P the tension in PQ is $g(x+l)$; after the wave reaches it the tension is gx. The potential energy given up by the element is, therefore,

$$\frac{g^2}{2\lambda}\{(x+l)^2-x^2\}dx=\frac{g^2}{2\lambda}(l^2+2lx)dx.$$

If $d\xi_1$ be the extension of the element in the first state and $d\xi_2$ in the second, the work done against gravity is

$$gx(d\xi_1-d\xi_2)$$

which is equal to $\frac{g^2lx}{\lambda}dx$,

since $d\xi_1=\frac{g(l+x)}{\lambda}dx$, and $d\xi_2=\frac{gx}{\lambda}dx$.

The kinetic energy acquired by the element may be put equal to $\frac{1}{2}v^2dx$, we have, therefore, the equation

$$\frac{1}{2}v^2dx=\frac{g^2}{2\lambda}(l^2+2lx)dx-\frac{g^2lx}{\lambda}dx;$$

therefore $v=\frac{gl}{\sqrt{\lambda}}=V.$

The motion until the wave reaches A will therefore be of the character stated above.

3. With regard to the *velocity of transmission of the wave*, we may investigate it in the following way.

While the wave is passing over PQ, the upper part of it is under a tension $g(l+x)$, while the tension at P is gx; the resultant force on PQ is therefore gl, and in the time dt this face generates momentum $gldt$. But the momentum generated in this time is equal to Vdx, hence we have

$$dx : dt = \frac{gl}{V} = \sqrt{\lambda}.$$

This is, therefore, the rate at which the wave passes along the particles of the string; and we see that, considered with reference to the unstretched string, it is uniform.

This result will apply to all the waves which we shall have to consider, and we need not, therefore, repeat it.

4. We have now to enquire what takes place when the wave reaches A. The highest element may be supposed to receive for a moment the velocity V, but to have its velocity instantly destroyed by impact at the fixed obstacle to which the string is attached. The other parts of the string will continue to move upwards until their velocity is destroyed by the pressure which is produced by the contraction of the string. What takes place is this: each particle of the string is reduced to rest in succession till it is all brought to rest, and, if we suppose $BC > AC$, the strain in the string will be the same as if CA were produced to B' (where $CB' = CB$), and the contraction proceeded uniformly from B', the stress at any point P being $= g.PB'$.

[If AC (fig. 7) were greater than BC we should still have to find a point B' by the same construction, and AB' would remain stretched during the return wave, while $B'PC$ would be contracted].

To verify the above statement, let the unstretched length of PC be x, and consider an element dx. Before and after the passage of the wave the stresses in dx are gx, and $g(l-x)$.

The kinetic energy taken up is therefore $\frac{g^2(l^2-2lx)}{2\lambda}$.

The work done against gravity is

$$\frac{gx}{\lambda}(x+l-x)\,dx = \frac{gxl}{\lambda}\,dx,$$

and the kinetic energy given up by the reduction of the element to rest is $\frac{1}{2}V^2dx$. These parts exactly counterbalance each other, and therefore the motion is of the kind described above.

5. For a moment the lowest element is in a state of contraction; but, the end being free, the tension is instantly relaxed, and it commences to shoot downwards, each particle receiving in succession the velocity V till the upgoing wave reaches A, and leaving the part of the string below it in the same state, as regards tension, as if it were hanging freely. The reader may readily verify this statement.

6. After the upgoing wave reaches A, the highest element, though momentarily in motion, is instantly reduced to rest, and a wave of rest runs down the string, which meanwhile continues to stretch until, when the wave reaches C, it is in exactly the same condition as when it was newly cut.

If (fig. 8) the unstretched lengths of CP, PQ be x, dx the tensions in PQ, before and after the passage of this wave, are gx and $g(l+x)$; and the potential energy acquired by it will be $\frac{g^2}{2\lambda}(l^2+2lx)$.

Gravity does an amount of work $=\frac{gx}{\lambda}(l+x-x)\,dx$, and the kinetic energy destroyed is $\frac{1}{2}V^2dx=\frac{1}{2}\frac{g^2l^2}{\lambda}dx$.

The principle of conservation of energy is therefore verified.

The string is now as it was when it was cut, and will continue to wriggle up and down through the same changes.

Motion of the falling part.

7. The motion of the falling part is even more interesting than that which we have just been considering, and may be investigated in somewhat the same way.

When first cut the tension of the element at C (fig. 9) is reduced from gl to zero, and therefore it acquires the velocity V, as we have seen already, for the other part AC.

A wave of V-velocity runs down the string, the lower part remaining, meanwhile, suspended at rest in space till the wave reaches it, and the stress in the upper part being that due to a contraction of the same absolute amount as if the string were inverted and hung freely from the point over which the wave is passing.

To verify these statements let the unstretched lengths of CP, PQ be as before x, dx, the energy surrendered by PQ

$$= \frac{g^2}{2\lambda} \{(l-x)^2 - x^2\} = \frac{g^2}{2\lambda} (l^2 - 2lx).$$

In addition, gravity does an amount of work upon CP in pulling it down which is equal to $\frac{gx}{\lambda} (l - x + x)\, dx = \frac{glx}{\lambda} dx$.

The kinetic energy acquired by PQ is therefore that due to a velocity V, where $V = \frac{gl}{\sqrt{\lambda}}$, the same as before.

8. When this wave of contraction reaches B, the contraction of the lowest element will be immediately relaxed, and it will commence to shoot out with a velocity $2V$, which will be communicated in succession to every element of the string, and the stress in the lowest part over which this upgoing wave has passed will be a tension equal to that due to the weight of the string below each point.

To show this, let the unstretched lengths of BP, PQ be x, dx, and let the contraction and extension of PQ before and after the passage of the wave be $d\xi_1$ and $d\xi_2$.

Assuming the distribution of stress to be that given above the energy surrendered by PQ during the passage of the wave will be

$$\frac{g^2}{2\lambda} \{(l-x)^2 - x^2\}\, dx = \frac{g^2}{2\lambda} (l^2 - 2lx)\, dx.$$

The work done by gravity on the string is of two parts, (1) that done on the whole string, considered as rigidly moving with the original velocity V, during the passage of the wave over PQ, (2) that due to the separation of the parts CQ, BP from each other. The velocity of CQ is V, and the time occupied by the wave in passing over $PQ = \frac{dx}{\sqrt{\lambda}}$, and hence the distance through which CQ falls in this time $= \frac{V dx}{\sqrt{(\lambda)}}$; the part (1) of the work due to gravity is thus equal to $\frac{glV}{\sqrt{\lambda}} dx = V^2 dx$.

Part (2)

$$= gx (d\xi_1 + d\xi_2) = \frac{gx}{\lambda} (l - x + x)\, dx = \frac{glx}{\lambda} dx,$$

as before.

If, therefore, V_1 be the new velocity acquired by PQ, the only change of kinetic energy experienced by the whole string during the passage of the wave over it will be

$$\tfrac{1}{2}(V_1^2 - V^2)\,dx = \frac{g^2}{2\lambda}(l^2 - 2lx)\,dx + V^2 dx + \frac{glx}{\lambda}\,dx,$$

whence we have at once

$$V_1 = 2V,$$

verifying the statement made above.

9. When this wave of extension reaches C it will be succeeded by a wave of contraction down the string corresponding to velocity $3V$, and thereafter by a series of wormlike movements whose velocities, beginning with the first, are V, $2V$, $3V$, $4V$, &c.

Suppose, for instance, that a wave of extension has just reached C, the velocity corresponding to it being nV, it will be followed by a wave of contraction corresponding to $(n+1)\,V$.

For let (fig. 11) the unstretched lengths of CP, PQ be x, dx, the stresses in PQ before and after the downward wave being numerically $g(l-x)$ and gx, the energy surrendered by PQ is $\frac{g^2}{2\lambda}(l^2 - 2lx)\,dx$.

The work done by gravity on the whole string is

(1) That on the string as a rigid body supposed to move with the velocity nV, and therefore $= gl.\frac{nV dx}{\sqrt{\lambda}}$.

(2) That due to the collapse of CP on BP, which is equal to $\frac{gx}{\lambda}(l - x + x)\,dx = \frac{glx}{\lambda}\,dx$.

The only change of kinetic energy which takes place in the string is that acquired by PQ, whose velocity changes from nV to V_{n+1}, suppose we have, from the equation of energy,

$$\tfrac{1}{2}(V_{n+1}^2 - n^2V^2)\,dx = \frac{g^2}{2\lambda}(l^2 - 2lx)\,dx + n\frac{gl}{\sqrt{\lambda}}\,V dx + \frac{glx}{\lambda}\,dx,$$

whence $$V_{n+1}^2 = n^2V^2 + 2nV^2 + V^2,$$

or $$V_{n+1} = (n+1)\,V.$$

The time which each of these separate movements requires for its accomplishment is $\frac{l}{\sqrt{\lambda}}$.

MATHEMATICAL NOTES.

On Long Successions of Composite Numbers.

Mr. Glaisher has given in the *Messenger* for November, 1877, and March, 1878, interesting results upon the sequences of composite numbers in the natural series. To the explanations given by the author, I add the following reflections upon the appearance of sequences much earlier than the theory indicates.

Denote by $N, = 2n+1$, any uneven number, by $P(q)$ the product of all the primes 2.3.5...9, q being the greatest prime inferior to n, and by $S_n(x)$ the series of the N consecutive numbers $xP(q)-n, \ldots xP(q), \ldots xP(q)+n$.

If we suppose $1<\alpha<n+1$, the number $xP(q)\pm\alpha$ is evidently composite, whatever integer value x may have.

Therefore, if $xP(q)-1$ and $xP(q)+1$ are composite numbers, the series $S_n(x)$ will be formed of N composite numbers. We have, for example, for $x=1$,

$$P(17)-1=61\times 8369, \qquad P(17)+1=19\times 97\times 277,$$

$$P(19)-1=53\times 197\times 929, \quad P(19)+1=347\times 27953,$$

whence the series

$$510492, \ldots\ 510510, \ldots\ 510528$$

contains 37 composite numbers, and the series

$$9699668, \ldots\ 9699690, \ldots\ 9699712$$

contains 45 consecutive composite numbers.

Thus, again, the series $S_{10}(x)$ contains 21 composite numbers for the values of x,

8, 15, 25, 26, 31, 33, 34, 37, 38, 45, 52, 54, 56, 58, 62, 71, 77, 79, 80, 82, 84, 91, 98, ...

In the general case, where q is given, we determine x by solving the two simultaneous congruences

$$xP(q)\equiv 1 \pmod{g} \quad \text{and} \quad xP(q)\equiv -1 \pmod{h},$$

g and h denoting the two primes that follow q.

EDOUARD LUCAS.

Paris, *August*, 1878.

The Intrinsic Equation of the Elastic Curve.

The intrinsic equation of the elastic curve or lintearia ($y\rho = c^2$) is

$$\psi = 2 \operatorname{am}\left(\frac{s}{kc}, k\right),$$

where $k = \frac{2c}{a}$, a being the greatest distance of the curve from the axis.

For
$$\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{2}{kc} \operatorname{dn} \frac{s}{kc},$$

and
$$\sin\psi = -\frac{dy}{ds} = \sin 2\operatorname{am} \frac{s}{kc}$$

$$= 2\operatorname{sn} \frac{s}{kc} \operatorname{cn} \frac{s}{kc};$$

therefore
$$y = \frac{2c}{k} \operatorname{dn} \frac{s}{kc};$$

and therefore $y\rho = c^2$.

If $k < 1$, ψ can increase indefinitely, and the curve is of the form given in fig. 34, Thomson and Tait, *Natural Philosophy*, § 611.

If $k = 1$, we have fig. 33.

If $k > 1$, ψ oscillates between the values $\pm \sin \frac{1}{k}$; and putting $k = \frac{1}{k_1}$, we have

$$y = a \operatorname{dn}\left(\frac{s}{kc}, k\right) = a \operatorname{cn}\left(\frac{s}{c}, k_1\right),$$

whence $k_1 = \frac{a}{2c}$, and the curve is of the form given in figures 28 to 32.

A. G. GREENHILL.

A Theorem in Combinations.

The number of combinations of n elements r at a time when every element may be repeated any number of times in the same combination, is

$$\frac{n(n+1)(n+2)\ldots(n+r-1)}{\lfloor r}.$$

I have given in "*Choice and Chance*," Prop. x. p. 67, (3rd edition), an algebraical proof of this proposition, dependent

only on first principles, and at p. 45 I have cast the same proof into the form of a proof of a rule of arithmetic.

But though that proof is very short, it is not very easy; and the following, though not so concise, may be preferred.

To fix the ideas, think of the n elements as being an alphabet of n letters; and observe that the number of permutations of any letters r at a time in a particular order, say alphabetical order, is the same as the number of combinations of the same letters r at a time. The problem is therefore equivalent to finding the number of r-lettered words that can be formed with an alphabet of n letters, each letter appearing as often as we please, but all the letters of the word being arranged in alphabetical order.

At equal distances draw n parallel lines αA, βB, ... ζZ (fig. 12) to represent the n letters of the alphabet, and at the same distances draw $r+1$ lines at right angles to them, so as to form a series of squares (for convenience, we will call then square inches), having $n-1$ squares in each horizontal row, and r squares in each vertical column. Then any r lettered word subject to the required conditions may be represented by proceeding vertically from α for as many inches as the number of times α is to occur in the word, thence proceeding horizontally to the line representing the next letter which comes into the word, and going down vertically as many inches as that letter is to occur, and so on, until we finish at the point Z. The number of words is therefore the same as the number of ways of proceeding from α to Z by a zigzag course, *i.e.* it is the number of orders in which $n-1$ horizontal steps and r vertical steps can be taken, or the number of ways in which $n+r-1$ steps can be taken, so that $n-1$ of them are horizontal and r are vertical, which is known to be

$$\frac{\lfloor n+r-1}{\lfloor r \lfloor n-1} \text{ or } \frac{n(n+1)(n+2)\ldots(n+r-1)}{\lfloor r}.$$

Therefore, &c. Q. E. D.

W. Allen Whitworth.

Generalisation of Prof. Cayley's theorem on partitions (vol. v. p. 188).

Prof. Cayley's theorem is, that if u_n denotes the number of partitions of n, no part less than 2 and order attended to (so that *e.g.* for $n=7$ the partitions are 7, 52, 25, 43, 34, 322, 232, 223), then $u_2=1$, $u_3=1$, $u_4=2$, $u_5=3$, $u_6=5$, $u_7=8$, $u_8=13$, &c., where each term is the sum of the preceding two terms.

The generalisation is, that if n_r denotes the number of partitions of n, no part less than r and order attended to, then $n_r = (n-1)_r + (n-r)_r$, *i.e.* each term = the next preceding term + the r^{th} preceding term. When $r=1$, so that all the numbers may occur in the partitions, we have $n_1 = 2(n-1)$, whence $n_1 = 2^{n-1}$, as is well known. The general formula is obtained by considering the coefficient of x^n in the expansion of $1 \div \{1 - (x^r + x^{r+1} + \&c.)\}$, $= (1-x) \div (1-x-x^r)$.

J. W. L. GLAISHER.

ON AMPÈRE'S ELECTRODYNAMIC THEORY.

By *T. C. Lewis, M.A.*, Fellow of Trinity College, Cambridge.

SOME of the results of Ampère's Theory, which are usually deduced by integration from his celebrated formula, seem to be capable of a much more elementary proof. The methods usually employed are all, except that by Quaternions, very long, and without exception they are complicated, and the meaning of the processes is masked under the difficulties of the analysis. It seemed likely that those results which admit of a simple verbal statement, might possibly admit of a correspondingly simple proof, based immediately on the experimental laws which are at the foundation of his analysis. The proofs given in this paper are (*a*) of the existence at every point of space of the "directrice de l'action électrodynamique;" (*b*) that the action on an element of current at any point is perpendicular to that element and to the 'directrice' or action at the point; (*c*) that the action on an element is proportional to the magnitude of the directrice, and to the sine of the angle included between it and the element. The formula itself cannot of course be proved without analysis, but, as it gives the effect of an isolated element of a circuit upon another element, it can only be practically useful when integrated, and some of the integrated results are what are here demonstrated without help from the formula.

Ampère's experimental laws are—

I. Equal and opposite currents in the same conductors produce equal and opposite effects on other conductors; whence it follows that an element of one current has no effect on an element of another which lies in a plane bisecting the former at right angles (and therefore is not affected by it).

II. The effect of a conductor bent or twisted in any

manner is equivalent to that of a straight one, provided that the two are traversed by equal currents, and the former nearly coincides with the latter.

III. No closed circuit can set in motion an element of a circular conductor about an axis through the centre of the circle and perpendicular to its plane.

IV. In similar systems traversed by equal currents the forces are equal.

To these we add the assumption, that the effect of any element of a current on another varies directly as the product of the strengths of the currents and of the lengths of the elements. It will not be necessary to assume that the action between two elements of currents is in the straight line joining them.

1. Since in nature electricity must flow in closed circuits, and therefore the results we wish to obtain are due to the whole circuit or circuits (and not to their isolated elements), we need not consider more than that part of the action of one element which does not necessarily vanish when summed up for all the elements of the circuits. Hence, in considering the action of one element on another, we may make use of the results observed when a whole circuit is acting instead. Now, in the latter case, we know by Ampère's second law that *for any single element* of a second circuit, we may substitute two others which do not deviate far from it, and which provide a road for the electricity to go from one end to the other of the element. Hence, in considering the effect of *one element* of the first circuit, we may also also resolve the given element of the second circuit into two parts, and consider the effect on both together to be the same as on that for which they were substituted; there may be some difference in these effects, but, at any rate, if this difference be summed up for the whole of the first circuit, it vanishes, and therefore need not be considered in this investigation. From Law III. it is inferred, that the effect of a closed circuit upon an element of a current, resolved in the direction of that element, is zero; therefore we need, as in the case of Law II., only consider that part of the effect, *even of an element* of the circuit, which is perpendicular to the element acted upon.

2. By Law I. the action of one element upon another vanishes, when the plane bisecting the latter at right angles contains the former. There is therefore one direction of the

element α_1 which is being acted upon, such that the action of any particular element of the system of current electricity upon it is zero, and this direction is perpendicular to the plane containing the influencing element and the middle point of α_1.

Let AB be this neutral position of α_1 for the action upon it of any chosen element of the influencing system. Take any other point C near AB, and join AC, BC. Then the actions on AC and CB, so far as they affect our results, are together equal to that on AB, which is zero; therefore the actions on BC and AC are in same straight line, but they are also perpendicular to BC and AC respectively. The only line which is perpendicular to both is the normal to the plane BAC, and is therefore perpendicular to AB. Thus the action on the element for any direction of the element in the plane ABC is in a constant direction; and for any direction whatsoever of the element, the action on it is perpendicular to AB, and therefore lies in a fixed plane; and is such, that it is perpendicular both to the actual and to the neutral position of α_1.

3. Let AD, AE be two positions of α_1 in the plane perpendicular to the neutral position. Let AD equal AE. Join DE. Then since the action on AD is the resultant of those on AE, ED by Law II. (if the same current flows by either path from A to D) resolve parallel to DE; then since all the actions we consider are perpendiculars to the elements on which they act, we see that the action on AD is equal to that on AE. Therefore if the element α_1 be rotated round its neutral position as axis, the action on it due to the corresponding element of the system of current electricity is unaltered.

4. It is now necessary to prove that there is a neutral position of α_1 under the action, not merely of an element of another current, but of any complete system of current electricity; and then the proofs of §§ 2 and 3 will become applicable in the general case. We have seen that, if the effect of one element only of the system of currents is considered, the neutral direction at any point is found by taking the plane containing the element and the point considered, and drawing a line perpendicular to the plane. Now by Law II. we may replace α_1, so far as it is necessary to consider the action of this one element of the closed circuit, by two pieces, one perpendicular to the above plane—on this the action is zero—and one parallel to the plane—on this the action is

proportional to its length, *i.e.* to $\alpha_1 \sin\theta$, where θ is the angle between the actual and neutral positions of α_1; it is also proportional to its intensity i, and to some function F, which depends upon the particular element of the system of current electricity whose effect we are considering, and upon the point from which α_1 is drawn, but (as we have proved in §§ 2 and 3) is independent of the direction in which α_1 is drawn. F is in fact the action of the element of the system in question per unit length of unit current placed along the direction of that part of α_1 which is alone affected by the element, *i.e.* the action per unit length of unit current in any direction perpendicular to the neutral direction, as proved in § 3. Now the action of the whole system upon α_1 is the sum of the actions of its parts. Divide the system up into very small elements, which may be supposed ultimately straight. Hence, if we suppose a number of forces F (corresponding to the various elements considered) to act along the corresponding neutral directions at α_1, $F\sin\theta$ will be the resolved part of any one of them perpendicular to α_1, and if we find the resultant of these, and turn it about the element α_1 through a right angle (which is of course equivalent to rotating the components, and then finding their resultant), we shall get the action upon a unit length of unit current placed along α_1 due to the whole system under consideration. We may distinguish any force F as the action at α_1 due to an element, and the resultant of the forces F as the action at α_1 due to the system of circuits. If the system of forces $F\sin\theta$ be in equilibrium, the resultant action *on* α_1 is zero; *i.e.* if α_1 be directed along the resultant of the system of forces F, the action on it is zero. *Hence, there is always a neutral direction for the element* α_1. The resultant of the system of forces F is the same as Ampère's "*Directrice de l'action électrodynamique.*"

5. Hence, as in §§ 2 and 3, we may prove that for any direction of the element α_1 the action of any system of electric currents on it is perpendicular to its actual and neutral directions, and hence it is always in a fixed plane, viz. that to which the neutral direction is normal. Further, if F_1 be the resultant of the system of forces F, θ_1 the angle between that resultant and α_1, we see, by means of § 4, that the action on α_1 is equal in magnitude to $i\alpha_1 F_1 \sin\theta$.

August, 1878.

TRANSACTIONS OF SOCIETIES.

The Meeting of the British Association at Dublin.

The forty-eighth meeting of the British Association for the Advancement of Science was opened at Dublin on Wednesday, August 14, 1878, under the Presidency of Mr. W. Spottiswoode, F.R.S. The greater part of the address of the President related to mathematics, his chief purpose being 'to inquire whether there be not points of contact in method or in subject-matter between mathematics and the outer world.' Three methods peculiar to modern mathematics, viz., 'first, that of Imaginary Quantities; secondly, that of Manifold Space; and, thirdly, that of Geometry not according to Euclid,' were explained and examined. The Sections assembled on Thursday morning, August 16, the following being the list of officers of Section *A* (Mathematical and Physical Science):

President.—Prof. G. Salmon, F.R.S.

Vice-Presidents.—Prof. R. S. Ball, F.R.S.; Prof. S. Haughton, F.R.S.; Prof. Henry Hennessey, F.R.S.; Dr. T. A. Hirst, F.R.S.; General Menabrea; Rev. Dr. Molloy; Prof. S. J. Perry, F.R.S.; Prof. John Purser; Prof. R. Townsend, F.R.S.; Prof. H. J S. Smith, F.R.S.; Mr. G. Johnstone Stoney, F.R.S.; Prof. J. J. Sylvester, F.R.S.; Sir William Thomson, F.R.S.

Secretaries.—Prof. John Casey, F.R.S.; Mr. G. F. Fitzgerald; Mr. J. W. L. Glaisher, F.R.S.; Mr. Oliver J. Lodge.

Among the mathematicians present at the meeting, besides those mentioned among the officers, were Prof. J. D. Everett; Mr. R. Harley, F.R.S.; Mr. W. M. Hicks; Mr. H. M. Jeffery; Prof. C. Niven; Mr. F. Purser; Prof. G. G. Stokes, F.R.S.; Mr. Benjamin Williamson.

Prof. Salmon was unable to be present at the meeting, having met with an accident, and a short inaugural address was given by Prof. Haughton. The following is the list of papers read in the department of mathematics on Monday, August 19, Prof. H. J. S. Smith in the chair:

C. W. Merrifield.—Report of the Committee on Babbage's Analytical Engine.

James Glaisher.—Report of the Committee on Mathematical Tables, with an Explanation of the Mode of Formation of the Factor Table for the Fourth Million.

J. W. L. Glaisher.—On certain special Enumerations of Primes.

J. W. L. Glaisher.—Notes on Circulating Decimals.

Prof. J. J. Sylvester.—On the Irreducible Derivatives of certain Binary Forms and Form-systems.

Prof. H. J. S. Smith.—On Quadric Transformation.

Prof. M. Falk.—Elementary Demonstration of the Theorem for the Multiplication of Determinants.

W. Spottiswoode.—On the "18 Coordinates" of a Conic in Space.

T. Archer Hirst.—On Halphen's new form of Chasles's Theorem on Systems of Conics satisfying four conditions.

Prof. Casey.—On Curves of the Third Order.

H. M. Jeffery.—On Cubics of the Third Class with three single Foci, both Plane and Spherical.

H. M. Jeffery.—On Cubic Surfaces referred to a Pentad of Cotangential Points.

Frederick Purser.—Note on the Geometrical treatment of Bicircular Quartics.

Prof. H. J. S. Smith.—On the Modular Curves.

Prof. Casey.—On a New Form of Tangential Equation.

Robert Harley.—On certain Linear Differential Equations.

J. W. L. Glaisher.—On the Solution of a Differential Equation connected with Riccati's.

Frederick Purser.—On the occurrence of Equal Roots in the Determinantal Equation of Small Oscillations.

J. W. L. Glaisher.—On the Law of Force to any point when the Orbit is a Conic.

Prof. R. S. Ball.—On the Principal Screws of Inertia of a free or constrained Rigid Body.

Prof. J. Purser.—On the applicability of Lagrange's Equations to certain Problems of Fluid Motion.

W. M. Hicks.—On the Motion of two Cylinders in a Fluid.

Besides these the following papers relating to mathematical subjects were read during the meeting:

Sir W. Thomson and Admiral Evans.—On the Tides of the Southern Hemisphere and Mediterranean.

Prof. S. Haughton.—On the Sun-heat received at the several Latitudes of the Earth, taking account of the Absorption of Heat by the Atmosphere, with conclusions as to the absolute Radiation of Earth-heat into Space, and the Minimum Duration of Geological Time.

G. H. Darwin.—On the Precession of a Viscous Spheroid.

G. F. Fitzgerald.—Note on Surface Tension.

Prof. J. Thomson.—On 'Dimensional Equations' and on some verbal expressions in Numerical Science.

The Babbage Committee reported that in their opinion the cost of completing Mr. Babbage's analytical engine would be expressed in tens of thousands of pounds at least, and they did not recommend the Association to take any steps to procure its construction. In the report of the Tables Committee it was stated that the factor table for the fourth million was completed and ready for press, and that the factor tables for the fifth and sixth millions were being actively proceeded with. A new Committee, consisting of Prof. Sylvester and Prof. Cayley, was appointed for the calculation of tables of the fundamental invariants of algebraic forms, and the sum of £50 was placed at their disposal for the purpose. The whole sum granted on the recommendation of Section *A* was £430.

The Association adjourned on August 21 till Wednesday, August 6, 1879, at Sheffield (not Nottingham, as decided at the Plymouth Meeting). Mr. G. J. Allman is the President elect. In 1880 the Meeting will be held at Swansea.

J. W. L. G.

FLUID MOTION IN A ROTATING QUADRANTAL CYLINDER.

By Professor *A. G. Greenhill, M.A.*

Mr. Hicks has found by summation (*ante*, pp. 42–44) the expression for ϕ, the velocity function of fluid motion in a rotating semicircular cylinder, bounded by $r=a$, and $\theta=0$, $\theta=\pi$, rotating about its axis with angular velocity ω; and the current function ψ, being the conjugate function to ϕ, can of course be immediately written down.

Generally, for liquid contained in a sectorial cylinder bounded by $r=a$ and $\theta=\pm\alpha$, rotating about its axis with angular velocity ω, it is well known that the current function

$$\psi=\tfrac{1}{2}\omega r^2\frac{\cos 2\theta}{\cos 2\alpha}$$

$$+32\omega a^2\alpha^2\Sigma_{n=0}^{n=\infty}(-1)^{n+1}\frac{\left(\frac{r}{a}\right)^{(2n+1)\frac{\pi}{2\alpha}}\cos(2n+1)\frac{\pi\theta}{2\alpha}}{\{(2n+1)\pi-4\alpha\}(2n+1)\pi\{(2n+1)\pi+4\alpha\}},$$

and therefore the velocity function

$$\phi = \tfrac{1}{2}\omega r^2 \frac{\sin 2\theta}{\cos 2\alpha}$$

$$+ 32\omega a^2\alpha^2 \Sigma_{n=0}^{n=\infty} (-1)^{n+1} \frac{\left(\frac{r}{a}\right)^{(2n+1)\frac{\pi}{2\alpha}} \sin(2n+1)\frac{\pi\theta}{2\alpha}}{\{(2n+1)\pi - 4\alpha\}(2n+1)\pi\{(2n+1)\pi + 4\alpha\}},$$

being the conjugate function to ψ.

For we must have

(1) $\nabla^2\psi = 0$, and ψ finite, within the sector.

(2) $\psi = \tfrac{1}{2}\omega r^2$, when $\theta = \pm\alpha$.

(3) $\psi = \tfrac{1}{2}\omega r^2$, when $r = a$.

Conditions (1) and (2) are satisfied by assuming

$$\psi = \tfrac{1}{2}\omega r^2 \frac{\cos 2\theta}{\cos 2\alpha} + \Sigma_{n=0}^{n=\infty} A_{2n+1} \left(\frac{r}{a}\right)^{(2n+1)\frac{\pi}{2\alpha}} \cos(2n+1)\frac{\pi\theta}{2\alpha},$$

and in order that condition (3) may be satisfied, namely $\psi = \tfrac{1}{2}\omega r^2$ when $r = a$, we must have

$$\Sigma_{n=0}^{n=\infty} A_{2n+1} \cos(2n+1)\frac{\pi\theta}{2\alpha} = \tfrac{1}{2}\omega a^2 \left(1 - \frac{\cos 2\theta}{\cos 2\alpha}\right),$$

between the limits $\theta = \pm\alpha$, and therefore by Fourier's theorem

$$A_{2n+1} = \frac{\omega a^2}{2\alpha} \int_0^{\alpha} \left(1 - \frac{\cos 2\theta}{\cos 2\alpha}\right) \cos(2n+1)\frac{\pi\theta}{2\alpha}\, d\theta$$

$$= \frac{\omega a^2}{2\alpha} \left[\frac{2\sin(2n+1)\frac{1}{2}\pi}{(2n+1)\frac{\pi}{2\alpha}} - \frac{\sin\{(2n+1)\frac{1}{2}\pi - 2\alpha\}}{\cos 2\alpha\,\{(2n+1)\frac{1}{2}\pi - 2\alpha\}} \right.$$

$$\left. - \frac{\sin\{(2n+1)\frac{1}{2}\pi + 2\alpha\}}{\cos 2\alpha\,\{(2n+1)\frac{1}{2}\pi + 2\alpha\}} \right]$$

$$= \omega a^2 (-1)^{n+1} \left\{ \frac{1}{(2n+1)\pi - 4\alpha} - \frac{2}{(2n+1)\pi} + \frac{1}{(2n+1)\pi + 4\alpha} \right\}$$

$$= 32\omega a^2\alpha^2 (-1)^{n+1} \frac{1}{\{(2n+1)\pi - 4\alpha\}(2n+1)\pi\{(2n+1)\pi + 4\alpha\}}.$$

Therefore

$$\psi + i\phi = \tfrac{1}{2}\omega \frac{r^2 e^{2i\theta}}{\cos 2\alpha}$$

$$+ 32\omega a^2\alpha^2 \Sigma_{n=0}^{n=\infty} \frac{(-1)^{n+1}\left(\frac{r}{a}e^{i\theta}\right)^{(2n+1)\frac{\pi}{2\alpha}}}{\{(2n+1)\pi - 4\alpha\}(2n+1)\pi\{(2n+1)\pi + 4\alpha\}}$$

$$= \tfrac{1}{2}\omega a^2 \frac{x^2}{\cos 2\alpha} + \frac{\omega a^2}{2\alpha} \Sigma_{n=0}^{n=\infty} (-1)^{n+1} x^{(2n+1)\frac{\pi}{2\alpha}}$$

$$\times \left\{ \frac{1}{(2n+1)\frac{\pi}{2\alpha} - 2} - \frac{2}{(2n+1)\frac{\pi}{2\alpha}} + \frac{1}{(2n+1)\frac{\pi}{2\alpha} + 2} \right\}$$

$$= \tfrac{1}{2}\omega a^2 \frac{x^2}{\cos 2\alpha}$$

$$- \frac{\omega a^2}{2\alpha}\left(x^2 \int_0 \frac{x^{\frac{\pi}{2\alpha}-3}\,dx}{1 + x^{\frac{\pi}{\alpha}}} - \frac{4\alpha}{\pi} \tan^{-1} x^{\frac{\pi}{2\alpha}} + x^{-2} \int_0 \frac{x^{\frac{\pi}{2\alpha}+1}\,dx}{1 + x^{\frac{\pi}{\alpha}}} \right),$$

putting $$x = \frac{r}{a} e^{i\theta}.$$

The integrations can be performed when $\frac{\pi}{2\alpha}$ is an integer, that is, when the angle of the sector is a submultiple of two right angles.

In a semi-circular cylinder $\frac{\pi}{2\alpha} = 1$, and we obtain the expression given by Mr. Hicks on p. 44.

In a quadrantal cylinder $\frac{\pi}{2\alpha} = 2$, and the first two terms in the series for $\psi + i\phi$ assume the indeterminate from $\infty - \infty$, and must be evaluated.

Putting in these terms $\frac{\pi}{2\alpha} = 2 + z$, and neglecting z^2,

$$\psi + i\phi = \frac{2\omega a^2}{\pi}\left\{ \frac{x^2}{z} - \frac{x^{2+z}}{z} + \tan^{-1} x^2 + \left(x^2 - \frac{1}{x^2}\right) \int_0 \frac{x^3\,dx}{1 + x^4} \right\}$$

$$= \frac{2\omega a^2}{\pi}\left\{ -x^2 \log x + \tan^{-1} x^2 + \tfrac{1}{4}\left(x^2 - \frac{1}{x^2}\right) \log (1 + x^4) \right\}$$

$$= -\frac{2\omega a^2}{\pi}\left(\frac{r^2}{a^2}\cos 2\theta + i\frac{r^2}{a^2}\sin 2\theta\right)\left(\log\frac{r}{a} + i\theta\right)$$

$$+\frac{\omega a^2}{\pi}\left\{\tan^{-1}\frac{2\frac{r^2}{a^2}\cos 2\theta}{1-\frac{r^4}{a^4}} + \tfrac{1}{2}i\log\frac{1+2\frac{r^2}{a^2}\sin 2\theta+\frac{r^4}{a^4}}{1-2\frac{r^2}{a^2}\sin 2\theta+\frac{r^4}{a^4}}\right\}$$

$$+\frac{\omega a^2}{2\pi}\left\{\left(\frac{r^2}{a^2}-\frac{a^2}{r^2}\right)\cos 2\theta + i\left(\frac{r^2}{a^2}+\frac{a^2}{r^2}\right)\sin 2\theta\right\}$$

$$\times\left\{\tfrac{1}{2}\log\left(1+2\frac{r^4}{a^4}\sin 4\theta+\frac{r^8}{a^8}\right) + i\tan^{-1}\frac{\frac{r^4}{a^4}\sin 4\theta}{1+\frac{r^4}{a^4}\cos 4\theta}\right\};$$

and therefore

$$\psi = -\frac{2\omega}{\pi}r^2\cos 2\theta\log\frac{r}{a} + \frac{2\omega}{\pi}r^2\sin 2\theta.\theta$$

$$+\frac{\omega a^2}{\pi}\tan^{-1}\frac{2\frac{r^2}{a^2}\cos 2\theta}{1-\frac{r^4}{a^4}}$$

$$+\frac{\omega a^2}{4\pi}\left(\frac{r^2}{a^2}-\frac{a^2}{r^2}\right)\cos 2\theta\log\left(1+2\frac{r^4}{a^4}\cos 4\theta+\frac{r^8}{a^8}\right)$$

$$-\frac{\omega a^2}{2\pi}\left(\frac{r^2}{a^2}+\frac{a^2}{r^2}\right)\sin 2\theta\tan^{-1}\frac{\frac{r^4}{a^4}\sin 4\theta}{1+\frac{r^4}{a^4}\cos 4\theta},$$

$$\phi = -\frac{2\omega}{\pi}r^2\cos 2\theta.\theta - \frac{2\omega}{\pi}r^2\sin 2\theta\log\frac{r}{a}$$

$$+\frac{\omega a^2}{2\pi}\log\frac{1+2\frac{r^2}{a^2}\sin 2\theta+\frac{r^4}{a^4}}{1-2\frac{r^2}{a^2}\sin 2\theta+\frac{r^4}{a^4}}$$

$$+\frac{\omega a^2}{2\pi}\left(\frac{r^2}{a^2}-\frac{a^2}{r^2}\right)\cos 2\theta \tan^{-1}\frac{\frac{r^4}{a^4}\sin 4\theta}{1+\frac{r^4}{a^4}\cos 4\theta}$$

$$+\frac{\omega a}{4\pi}\left(\frac{r^2}{a^2}+\frac{a^2}{r^2}\right)\sin 2\theta \log\left(1+2\frac{r^4}{a^4}\cos 4\theta\frac{r^8}{a^4}\right).$$

It is easily seen that $\nabla^2\psi=0$ and $\nabla^2\phi=0$ from the consideration that if x_1 and y_1, and also x_2 and y_2 are conjugate functions, then $x_1x_2-y_1y_2$ and $x_1y_2+x_2y_1$ are conjugate functions (Maxwell, *Electricity*, § 187), and we may put

$$x_1=r^2\cos 2\theta,\quad y_1=r^2\sin 2\theta,$$

$$x_2=\log\frac{r}{a},\qquad y_2=\theta;$$

also $$x_1=\left(\frac{r^2}{a^2}-\frac{a^2}{r^2}\right)\cos 2\theta,\quad y_1=\left(\frac{r^2}{a^2}+\frac{a^2}{r^2}\right)\sin 2\theta,$$

$$x_2=\tfrac{1}{2}\log\left(1+2\frac{r^4}{a_4}\cos^4\theta+\frac{r^8}{a^8}\right),\quad y_2=\tan^{-1}\frac{\frac{r^4}{a^4}\sin 4\theta}{1+\frac{r^4}{a^4}\cos 4\theta},$$

and then $x_1x_2-y_1y_2$ and $x_1y_2+x_2y_1$ form component parts of ψ and ϕ.

Also when $r=a$,

$$\psi=\frac{2\omega a^2}{\pi}\sin 2\theta\,.\,\theta-\frac{\omega a^2}{\pi}\sin 2\theta\tan^{-1}\tan 2\theta+\frac{\omega a^2}{\pi}\tan^{-1}\infty$$

$$=\tfrac{1}{2}\omega a^2,$$

and when $$\theta=\pm\tfrac{1}{4}\pi,\quad \psi=\tfrac{1}{2}\omega r^2,$$

and therefore ψ and ϕ satisfy the required conditions.

If we put $\psi=\chi+\frac{1}{2}\omega r^2$, then within the sector $\nabla^2\chi=-2\omega$, and $\chi=0$ round the boundary; χ is the current function of the liquid relative to the cylinder, and $\chi=C$ is the equation of the curve described by a particle of liquid relative to the cylinder.

If a sectorial cylinder filled with liquid perform finite oscillations under gravity about its axis which is held fixed, it will oscillate like a simple circular pendulum, and to find the length of the simple equivalent pendulum, we require to know the kinetic energy of the liquid due to a given angular velocity of the cylinder about the axis.

If T denote the kinetic energy per unit length of the cylinder due to the angular velocity ω, and if ϕ denote the velocity function, ρ the density of the liquid,

$$T = \tfrac{1}{2}\rho \int \left\{ \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 \right\} dA$$

$$= \tfrac{1}{2}\rho \int \phi \frac{d\phi}{dn}\, ds,$$

by Green's theorem.

For a sector bounded by the arc $r = a$, and the radii $\theta = \alpha$ and $\theta = -\alpha$, $\dfrac{d\phi}{dn} = 0$ when $r = a$, and $\dfrac{d\phi}{dn} = \pm \omega r$ when $\theta = \pm\alpha$; also $\phi_\alpha = -\phi_{-\alpha}$.

Therefore
$$T = \tfrac{1}{2}\omega\rho \int_0^a (\phi_\alpha - \phi_{-\alpha})\, r dr$$

$$= \omega\rho \int_0^a \phi_\alpha r dr.$$

For a semi-circular sector bounded by the radii $\theta = 0$ and $\theta = \pi$,

$$\phi = \tfrac{1}{2}\omega r^2 \sin 2\theta$$

$$+ \frac{\omega a^2}{4\pi} \left[\left\{ \left(\frac{r^2}{a^2} + \frac{a^2}{r^2}\right) \cos 2\theta - 2 \right\} \log \frac{1 + 2\dfrac{r}{a}\cos\theta + \dfrac{r^2}{a^2}}{1 - 2\dfrac{r}{a}\cos\theta + \dfrac{r^2}{a^2}} \right.$$

$$\left. - 2\left(\frac{r^2}{a^2} - \frac{a^2}{r^2}\right) \sin 2\theta \tan^{-1} \frac{2ar \sin\theta}{a^2 - r^2} \right] - \frac{\omega a^2}{\pi} \left(\frac{r}{a} + \frac{a}{r}\right) \cos\theta,$$

and, therefore,

$$\phi_\pi = \frac{\omega a^2}{2\pi} \left(\frac{r}{a} - \frac{a}{r}\right)^2 \log \frac{a - r}{a + r} + \frac{\omega a^2}{\pi} \left(\frac{r}{a} + \frac{a}{r}\right) = -\phi_0.$$

Therefore
$$T = \omega\rho \int_0^a \phi_\pi r dr$$

$$= \frac{\rho\omega^2 a^2}{2\pi} \int_0^a \left\{ \left(\frac{r}{a} - \frac{a}{r}\right)^2 \log \frac{a - r}{a + r} + 2\left(\frac{r}{a} + \frac{a}{r}\right) \right\} r dr,$$

and putting $\frac{r}{a} = z,$

$$T = \tfrac{1}{2}\rho\omega^2 \frac{a^4}{2\pi} \int_0^1 \left\{\left(z^3 - 2z + \frac{1}{z}\right) \log \frac{1-z}{1+z} + 2z^2 + 2\right\} dz$$

$$= \rho\omega^2 \frac{a^4}{2\pi} (4 - \tfrac{1}{4}\pi^2).$$

Denoting the mass of the liquid per unit length of the cylinder by M, and the effective radius of gyration of the liquid in the semi-circular sector by k, we have

$$M = \tfrac{1}{2}\rho\pi a^2,$$

$$\tfrac{1}{2}Mk^2\omega^2 = T = \tfrac{1}{2}\rho\omega^2 \frac{a^4}{\pi} (4 - \tfrac{1}{4}\pi^2),$$

and therefore $$k^2 = \frac{8a^2}{\pi^2} - \tfrac{1}{4}a^2.$$

If the liquid be solidified, the effective radius of gyration of the liquid will be given by

$$k'^2 = \tfrac{1}{2}a^2,$$

and therefore $$\frac{k^2}{k'^2} = \frac{16}{\pi^2} - 1 = \frac{l}{l'},$$

the ratio of the lengths of the equivalent pendulums, when the semi-circular sector is filled with liquid, and when the liquid is solidified.

In a quadrantal sector, bounded by the radii $\theta = \pm\frac{1}{4}\pi$,

$$\phi = -\frac{2\omega}{\pi} r^2 \cos 2\theta \,.\, \theta - \frac{2\omega}{\pi} r^2 \sin 2\theta \log \frac{r}{a}$$

$$+ \tfrac{1}{4} \frac{\omega a^2}{\pi} \left(\frac{r^2}{a^2} + \frac{a^2}{r^2}\right) \sin 2\theta \log \left(1 + 2\frac{r^4}{a^4} \cos 4\theta + \frac{r^8}{a^8}\right)$$

$$+ \tfrac{1}{2} \frac{\omega a^2}{\pi} \left(\frac{r^2}{a^2} - \frac{a^2}{r^2}\right) \cos 2\theta \tan^{-1} \frac{r^4 \sin 4\theta}{a^4 + r^4 \cos 4\theta}$$

$$+ \tfrac{1}{2} \frac{\omega a^2}{\pi} \log \frac{1 + 2\frac{r^2}{a^2} \sin 2\theta + \frac{r^4}{a^4}}{1 - 2\frac{r}{a^2} \sin 2\theta + \frac{r^4}{a^4}};$$

and therefore

$$\phi_{\frac{1}{4}\pi} = -\frac{2\omega}{\pi} r^2 \log\frac{r}{a} + \tfrac{1}{2}\frac{\omega a^2}{\pi}\left(\frac{r^2}{a^2}+\frac{a^2}{r^2}\right)\log\left(1-\frac{r^4}{a^4}\right) + \frac{\omega a^2}{\pi}\log\frac{1+\frac{r^2}{a^2}}{1-\frac{r^2}{a^2}},$$

and $$T = \rho\omega\int_0^a \phi_{\frac{1}{4}\pi}\, r\, dr$$

$$= \tfrac{1}{2}\rho\omega^2\frac{a^4}{\pi}\int_0^1\left\{-z\log z + \tfrac{1}{2}\left(z+\frac{1}{z}\right)\log(1-z^2) + \log\frac{1+z}{1-z}\right\}dz,$$

if $$z = \frac{r^2}{a^2}.$$

Performing the integration, we shall find

$$T = \rho\omega^2\frac{a^4}{\pi}\left(\log 2 - \frac{\pi^2}{48}\right),$$

and therefore if k denote the effective radius of gyration of the liquid in a quadrantal sector about the axis,

$$k^2 = \frac{T}{\frac{1}{2}M\omega^2} = \frac{\rho\omega^2\frac{a^4}{\pi}\left(\log 2 - \frac{\pi^2}{48}\right)}{\frac{1}{8}\rho\omega^2\pi a^2}$$

$$= \frac{8a^2}{\pi^2}\log 2 - \tfrac{1}{6}a^2,$$

and $$\frac{k^2}{k'^2} = \frac{16}{\pi^2}\log 2 - \tfrac{1}{3} = \frac{l}{l'},$$

the ratio of the lengths of the simple equivalent pendulums, when the quadrantal sector is filled with liquid, and when the liquid is solidified.

For a sector of 60°, $\frac{\pi}{2\alpha} = 3$, and therefore

$$\psi + i\phi = \omega a^2 x^2 + \frac{2\omega a^2}{\pi}\tan^{-1}x^3$$

$$- \frac{3\omega a^2}{\pi}\left(x^2\int_0\frac{dx}{1+x^6} + x^{-2}\int_0\frac{x^7 dx}{1+x^6}\right)$$

$$= \omega a^2 x^2 + \frac{2\omega a^2}{\pi} \tan^{-1} x^3$$

$$- \frac{\omega a^2}{\pi} \left(x^2 - \frac{1}{x^2}\right) \frac{\sqrt{3}}{2} \log \frac{x^2 + x\sqrt{3} + 1}{x^2 - x\sqrt{3} + 1}$$

$$- \frac{\omega a^2}{\pi} \left(x^2 + \frac{1}{x^2}\right) \left(\tan^{-1} \frac{x}{1 - x^2} + 2 \tan^{-1} x\right)$$

$$= \omega r^2 \cos 2\theta + i\omega r^2 \sin 2\theta$$

$$+ \frac{\omega a^2}{\pi} \left(\tan^{-1} \frac{2\frac{r^3}{a^3} \cos 3\theta}{1 - \frac{r^6}{a^6}} + \tfrac{1}{2} i \log \frac{1 + 2\frac{r^3}{a^3} \sin 3\theta + \frac{r^6}{a^6}}{1 - 2\frac{r^3}{a^3} \sin 3\theta + \frac{r^6}{a^6}}\right)$$

$$- \frac{\sqrt{3}}{2} \frac{\omega a^2}{\pi} \left\{\left(\frac{r^2}{a^2} - \frac{a^2}{r^2}\right) \cos 2\theta + i \left(\frac{r^2}{a^2} + \frac{a^2}{r^2}\right) \sin 2\theta\right\}$$

$$\times \left\{\tfrac{1}{2} \log \frac{1 + 2\sqrt{3}\frac{r}{a} \cos\theta + \frac{r^2}{a^2}(2\cos 2\theta + 3) + 2\sqrt{3}\frac{r^3}{a^3}\cos\theta + \frac{r^4}{a^4}}{1 - 2\sqrt{3}\frac{r}{a}\cos\theta + \frac{r^2}{a^2}(2\cos 2\theta + 3) - 2\sqrt{3}\frac{r^3}{a^3}\cos\theta + \frac{r^4}{a^4}}\right.$$

$$\left. + i \tan^{-1} \frac{2\sqrt{3}\frac{r}{a}\sin\theta \left(1 - \frac{r^2}{a^2}\right)}{1 + \frac{r^2}{a^2}(2\cos 2\theta - 3) + \frac{r^4}{a^4}}\right\}$$

$$- \frac{\omega a^2}{\pi} \left\{\left(\frac{r^2}{a^2} + \frac{a^2}{r^2}\right) \cos 2\theta + i \left(\frac{r^2}{a^2} - \frac{a^2}{r^2}\right) \sin 2\theta\right\}$$

$$\times \left\{\tfrac{1}{2} \tan^{-1} \frac{2\frac{r}{a}\cos\theta \left(1 - \frac{r^2}{a^2}\right)}{1 - (1 + 2\cos 2\theta)\frac{r^2}{a^2} + \frac{r^4}{a^4}}\right.$$

$$+ \tfrac{1}{4} i \log \frac{1 + 2\frac{r}{a}\sin\theta + \frac{r^2}{a^2}(1 - 2\cos 2\theta) + 2\frac{r^3}{a^3}\sin\theta + \frac{r^4}{a^4}}{1 - 2\frac{r}{a}\sin\theta + \frac{r^2}{a^2}(1 - 2\cos 2\theta) - 2\frac{r^3}{a^3}\sin\theta + \frac{r^4}{a^4}}$$

$$\left. + \tan^{-1} \frac{2\frac{r}{a}\cos\theta}{1 - \frac{r^2}{a^2}} + \tfrac{1}{2} i \log \frac{1 + 2\frac{r}{a}\sin\theta + \frac{r^2}{a^2}}{1 - 2\frac{r}{a}\sin\theta + \frac{r^2}{a^2}}\right\},$$

using the theorem that

$$\log(\alpha+i\beta)=\tfrac{1}{2}\log(\alpha^2+\beta^2)+i\tan$$

and $$\tan^{-1}x=\tfrac{1}{2}i\log\frac{1-ix}{1+ix};$$

and therefore

$$\tan^{-1}(\alpha+i\beta)=\tfrac{1}{2}i\log\frac{1+\beta-i\alpha}{1-\beta+i\alpha}$$

$$=\tfrac{1}{4}i\log\frac{(1+\beta)^2+\alpha^2}{(1-\beta)^2+\alpha^2}+\tfrac{1}{2}\tan^{-1}\frac{\alpha}{1+\beta}+$$

$$=\tfrac{1}{2}\tan^{-1}\frac{2\alpha}{1-\alpha^2-\beta^2}+\tfrac{1}{4}i\log\frac{(1+\beta)^2+}{(1-\beta)^2+}$$

Therefore, equating the real and imagi

$$\psi=\omega r^2\cos 2\theta+\frac{\omega a^2}{\pi}\tan^{-1}\frac{2\frac{r^3}{a^3}\cos 3\theta}{1-\frac{r^6}{a^6}}$$

$$-\frac{\sqrt{3}}{4}\frac{\omega a^2}{\pi}\left(\frac{r^2}{a^2}-\frac{a^2}{r^2}\right)\cos 2\theta$$

$$\times\log\frac{1+2\sqrt{3}\frac{r}{a}\cos\theta+\frac{r^2}{a^2}(2\cos 2\theta+3)+}{1-2\sqrt{3}\frac{r}{a}\cos\theta+\frac{r^2}{a^2}(2\cos 2\theta+3)-}$$

$$+\frac{\sqrt{3}}{2}\frac{\omega a^2}{\pi}\left(\frac{r^2}{a^2}+\frac{a^2}{r^2}\right)\sin 2\theta\tan^{-1}\frac{2\sqrt{3}\frac{r}{a}s}{1+\frac{r^2}{a^2}(2c}$$

$$-\tfrac{1}{2}\frac{\omega a^2}{\pi}\left(\frac{r^2}{a^2}+\frac{a^2}{r^2}\right)\cos 2\theta$$

$$\times\left\{\tan^{-1}\frac{2\frac{r}{a}\cos\theta\left(1-\frac{r^2}{a^2}\right)}{1-\frac{r^2}{a^2}(1+2\cos 2\theta)+\frac{r^4}{a^4}}+2\tan\right.$$

$$+\tfrac{1}{2}\frac{\omega a^2}{\pi}\left(\frac{r^2}{a^2}-\frac{a^2}{r^2}\right)\sin 2\theta$$

$$\left(\frac{1+\frac{r}{a}+\frac{r^2}{a^2}}{1-\frac{r}{a}+\frac{r^2}{a^2}}\right)^2$$

$$\tan^{-1}\frac{\sqrt{3}\,\frac{r}{a}}{1-\frac{r^2}{a^2}}$$

$$\left(\frac{1+\frac{r}{a}}{1-\frac{r}{a}}\right)^4\left(\frac{1-\frac{r}{a}+\frac{r^2}{a^2}}{1+\frac{r}{a}+\frac{r^2}{a^2}}\right)$$

$$\tan^{-1}\frac{\sqrt{3}\,\frac{r}{a}}{1-\frac{r^2}{a^2}};$$

us of gyration of the liquid,

$$\omega^2 = \frac{2\omega\rho\int_0^a \phi r dr}{\frac{1}{6}\pi\rho a^4\omega^2}$$

$$= \frac{12}{\pi a^4\omega}\int_0^a \phi_a r dr.$$

$$\frac{\pi}{2\alpha} = n, \text{ an integer}$$

$$a^2\frac{x^2}{\cos\frac{\pi}{n}} + 2\,\frac{\omega a^2}{\pi}\tan^{-1}x^n$$

$$-\frac{n\omega a^2}{\pi}\left(x^2\int\frac{x^{n-3}dx}{1+x^{2n}} + x^{-2}\int\frac{x^{n+1}dx}{1+x^{2n}}\right).$$

$$\times \left(\frac{1 + 2\frac{r}{a}\sin\theta + \frac{r^2}{a^2}}{1 - 2\frac{r}{a}\sin\theta + \frac{r^2}{a^2}} \right)^2 .$$

Obviously $\nabla^2\psi = 0$, and $\nabla^2\phi = 0$, and ψ and ϕ are finite inside the sector; also

(1) when $r = a$,

$$\psi = \omega a^2 \cos 2\theta + \frac{\omega a^2}{\pi} \tan^{-1}\infty - 2\frac{\omega a^2}{\pi}\cos 2\theta \tan^{-1}\infty$$
$$= \tfrac{1}{2}\omega a^2 ;$$

(2) when $\theta = \frac{1}{6}\pi$,

$$\psi = \tfrac{1}{2}\omega r^2$$
$$- \frac{\sqrt{3}}{8}\frac{\omega a^2}{\pi}\left(\frac{r^2}{a^2} - \frac{a^2}{r^2}\right) \log \left(\frac{1 + 3\frac{r}{a} + 4\frac{r^2}{a^2} + 3\frac{r^3}{a^3} + \frac{r^4}{a^4}}{1 - 3\frac{r}{a} + 4\frac{r^2}{a^2} - 3\frac{r^3}{a^3} + \frac{r^4}{a^4}} \right)$$
$$\times \left(\frac{1 - \frac{r}{a} - \frac{r^3}{a^3} + \frac{r^4}{a^4}}{1 + \frac{r}{a} + \frac{r^3}{a^3} + \frac{r^4}{a^4}} \right) \left(\frac{1 - \frac{r}{a} + \frac{r^2}{a^2}}{1 + \frac{r}{a} + \frac{r^2}{a^2}} \right)^2 .$$
$$+ \tfrac{1}{4}\frac{\omega a^2}{\pi}\left(\frac{r^2}{a^2} + \frac{a^2}{r^2}\right)\left(3\tan^{-1}\frac{\sqrt{3}\frac{r}{a}}{1 - \frac{r^2}{a^2}} - \tan^{-1}\frac{\sqrt{3}\frac{r}{a}}{1 - \frac{r^2}{a^2}} - 2\tan^{-1}\frac{\sqrt{3}\frac{r}{a}}{1 - \frac{r^2}{a^2}} \right)$$
$$= \tfrac{1}{2}\omega r^2$$

and so also when $\theta = -\frac{1}{6}\pi$, $\psi = \frac{1}{2}\omega r^2$; therefore ψ satisfies the required conditions.

Also

$$\phi_a = \frac{\sqrt{3}}{2}\omega r^2 + \frac{\omega a^2}{\pi}\log\frac{1 + \frac{r^3}{a^3}}{1 - \frac{r^3}{a^3}}$$
$$- \tfrac{1}{8}\frac{\omega a^2}{\pi}\left(\frac{r^2}{a^2} + \frac{a^2}{r^2}\right)\log\left(\frac{1 + 3\frac{r}{a} + 4\frac{r^2}{a^2} - 3\frac{r^3}{a^3} + \frac{r^4}{a^4}}{1 - 3\frac{r}{a} + 4\frac{r^2}{a^2} - 3\frac{r^3}{a^3} + \frac{r^4}{a^4}} \right)^2$$

$$\times \left(\frac{1+\frac{r}{a}+\frac{r^3}{a^3}+\frac{r^4}{a^4}}{1-\frac{r}{a}-\frac{r^3}{a^3}+\frac{r^4}{a^4}}\right)\left(\frac{1+\frac{r}{a}+\frac{r^2}{a^2}}{1-\frac{r}{a}+\frac{r^2}{a^2}}\right)^2$$

$$-\frac{\sqrt{3}}{2}\frac{\omega a^2}{\pi}\left(\frac{r^2}{a^2}-\frac{a^2}{r^2}\right)\tan^{-1}\frac{\sqrt{3}\,\frac{r}{a}}{1-\frac{r^2}{a^2}}$$

$$=\frac{\sqrt{3}}{2}\omega r^2+\frac{\omega a^2}{\pi}\log\frac{1+\frac{r^3}{a^3}}{1-\frac{r^3}{a^3}}$$

$$-\tfrac{1}{4}\frac{\omega a^2}{\pi}\left(\frac{r^2}{a^2}+\frac{a^2}{r^2}\right)\log\left(\frac{1+\frac{r}{a}}{1-\frac{r}{a}}\right)^4\left(\frac{1-\frac{r}{a}+\frac{r^2}{a^2}}{1+\frac{r}{a}+\frac{r^2}{a^2}}\right)$$

$$-\frac{\sqrt{3}}{2}\frac{\omega a^2}{\pi}\left(\frac{r^2}{a^2}-\frac{a^2}{r^2}\right)\tan^{-1}\frac{\sqrt{3}\,\frac{r}{a}}{1-\frac{r^2}{a^2}};$$

and k denoting the effective radius of gyration of the liquid,

$$k^2=\frac{2T}{M\omega^2}=\frac{2\omega\rho\int_0^a\phi r dr}{\frac{1}{8}\pi\rho a^4\omega^2}$$

$$=\frac{12}{\pi a^4\omega}\int_0^a\phi_a r dr.$$

Generally, if $\frac{\pi}{2\alpha}=n$, an integer

$$\psi+i\phi=\tfrac{1}{2}\omega a^2\frac{x^2}{\cos\frac{\pi}{n}}+2\frac{\omega a^2}{\pi}\tan^{-1}x^n$$

$$-\frac{n\omega a^2}{\pi}\left(x^2\int\frac{x^{n-3}dx}{1+x^{2n}}+x^{-2}\int\frac{x^{n+1}dx}{1+x^{2n}}\right).$$

Now $\int \frac{x^{n-3}dx}{1+x^{2n}}$ $(= A$ suppose)

$$= -\frac{1}{2n}\Sigma \cos(n-2)\frac{s\pi}{2n}\log\left(x^2 - 2x\cos\frac{s\pi}{2n}+1\right)$$

$$+\frac{1}{n}\Sigma \sin(n-2)\frac{s\pi}{2n}\tan^{-1}\frac{x-\cos\frac{s\pi}{2n}}{\sin\frac{s\pi}{2n}},$$

and $\int \frac{x^{n+1}dn}{1+x^{2n}}$ $(= B$ suppose)

$$= -\frac{1}{2n}\Sigma \cos(n+2)\frac{s\pi}{2n}\log\left(x^2 - 2x\cos\frac{s\pi}{2n}+1\right)$$

$$+\frac{1}{n}\Sigma \sin(n+2)\frac{s\pi}{2n}\tan^{-1}\frac{x-\cos\frac{s\pi}{2n}}{\sin\frac{s\pi}{2n}},$$

where Σ means a sum to be formed by giving to s all the odd integral values from 1 to $2n-1$ inclusive.

Therefore,

$$\psi + i\phi = \tfrac{1}{2}\omega a^2 \frac{x^2}{\cos\frac{\pi}{n}} + 2\frac{\omega a^2}{\pi}\tan^{-1}x^n$$

$$-\frac{n\omega a^2}{2\pi}\left\{\left(x^2 - \frac{1}{x^2}\right)(A-B) + \left(x^2+\frac{1}{x^2}\right)(A+B)\right\}$$

$$= \tfrac{1}{2}\omega a^2 \frac{x^2}{\cos\frac{\pi}{n}} + 2\frac{\omega a^2}{\pi}\tan^{-1}x^n$$

$$+\frac{\omega a^2}{2\pi}\left(x^2 - \frac{1}{x^2}\right)\Sigma \sin\frac{s\pi}{2}\sin\frac{s\pi}{n}\log\left(x^2 - 2x\cos\frac{s\pi}{2n}+1\right)$$

$$-\frac{\omega a^2}{\pi}\left(x^2+\frac{1}{x^2}\right)\Sigma \sin\frac{s\pi}{2}\cos\frac{s\pi}{n}\tan^{-1}\frac{x-\cos\frac{s\pi}{2n}}{\sin\frac{s\pi}{2n}}$$

$$= \tfrac{1}{2}\omega \frac{r^2\cos 2\theta}{\cos\frac{\pi}{n}} + \tfrac{1}{2}\omega \frac{r^2\sin 2\theta}{\cos\frac{\pi}{n}}$$

$$+\frac{\omega a^2}{\pi}\tan^{-1}\frac{2\frac{r^n}{a^n}\cos n\theta}{1-\frac{r^{2n}}{a^{2n}}}+\tfrac{1}{2}i\frac{\omega a^2}{\pi}\log\frac{1+2\frac{r^n}{a^n}\sin n\theta+\frac{r^{2n}}{a^{2n}}}{1-2\frac{r^n}{a^n}\sin n\theta+\frac{r^{2n}}{a^{2n}}}$$

$$+\frac{\omega a^2}{2\pi}\left\{\left(\frac{r^2}{a^2}-\frac{a^2}{r^2}\right)\cos 2\theta+i\left(\frac{r^2}{a^2}+\frac{a^2}{r^2}\right)\sin 2\theta\right\},$$

$$\Sigma\sin\frac{s\pi}{2}\sin\frac{s\pi}{n}$$

$$\times\left[\tfrac{1}{2}\log\left\{1-4\frac{r}{a}\cos\theta\cos\frac{s\pi}{2n}+\frac{r^2}{a^2}\left(\cos 2\theta-2\cos^2\frac{s\pi}{2n}\right)\right.\right.$$

$$\left.-4\frac{r^3}{a^3}\cos\theta\cos\frac{s\pi}{2n}+\frac{r^4}{a^4}\right\}$$

$$\left.-i\tan^{-1}\frac{2\frac{r}{a}\sin\theta\cos\frac{s\pi}{2n}-\frac{r^2}{a^2}\sin 2\theta}{1-2\frac{r}{a}\cos\theta\cos\frac{s\pi}{2n}+\frac{r^2}{a^2}\cos 2\theta}\right]$$

$$-\frac{\omega a^2}{\pi}\left\{\left(\frac{r^2}{a^2}+\frac{a^2}{r^2}\right)\cos 2\theta+i\left(\frac{r^2}{a^2}-\frac{a^2}{r^2}\right)\sin 2\theta\right\}$$

$$\Sigma\sin\frac{s\pi}{2}\cos\frac{s\pi}{n}\left[\tfrac{1}{2}\tan^{-1}\frac{\sin\frac{s\pi}{2n}\left(\cos\frac{s\pi}{2n}-\frac{r}{a}\cos\theta\right)}{\cos\frac{s\pi}{n}-2\frac{r}{a}\cos\theta\cos\frac{s\pi}{2n}+\frac{r^2}{a^2}}\right.$$

$$\left.+\tfrac{1}{4}i\log\frac{1-2\frac{r}{a}\cos\left(\frac{s\pi}{2n}-\theta\right)+\frac{r^2}{a^2}}{1+2\frac{r}{a}\cos\left(\frac{s\pi}{2n}+\theta\right)+\frac{r^2}{a^2}}\right]$$

and, therefore,

$$\psi=\tfrac{1}{2}\omega r^2\frac{\cos 2\theta}{\cos\frac{\pi}{n}}+\frac{\omega a^2}{\pi}\tan^{-1}\frac{2\frac{r^n}{a^n}\cos n\theta}{1-\frac{r^{2n}}{a^{2n}}}$$

$$+\frac{\omega a^2}{4\pi}\left(\frac{r^2}{a^2}-\frac{a^2}{r^2}\right)\cos 2\theta\,\Sigma\sin\frac{s\pi}{2}\sin\frac{s\pi}{n}$$

$$\times\log\left\{1-4\frac{r}{a}\cos\theta\cos\frac{s\pi}{2n}+2\frac{r^2}{a^2}\left(\cos 2\theta-2\cos^2\frac{s\pi}{2n}\right)\right.$$

$$- 4\frac{r^2}{a^2}\cos\theta\cos\frac{3\pi}{2n} + \frac{r^4}{a^4}\Big\}$$

$$+ \frac{\omega a^2}{2\pi}\left(\frac{r^2}{a^2} + \frac{a^2}{r^2}\right)\sin 2\theta\Sigma \sin\frac{3\pi}{2}\sin\frac{3\pi}{n}$$

$$\times \tan^{-1}\frac{2\frac{r}{a}\sin\theta\cos\frac{3\pi}{2n} - \frac{r^2}{a^2}\sin 2\theta}{1 - 2\frac{r}{a}\cos\theta\cos\frac{3\pi}{2n} + \frac{r^2}{a^2}\cos 2\theta}$$

$$- \frac{\omega a^2}{2\pi}\left(\frac{r^2}{a^2} + \frac{a^2}{r^2}\right)\cos 2\theta\Sigma \sin\frac{3\pi}{2}\sin\frac{3\pi}{n}$$

$$\times \tan^{-1}\frac{2\sin\frac{3\pi}{2n}\left(\cos\frac{3\pi}{2n} - \frac{r}{a}\cos\theta\right)}{\cos\frac{3\pi}{n} - 2\frac{r}{a}\cos\theta\cos\frac{3\pi}{n} + \frac{r^2}{a^2}}$$

$$+ \frac{\omega a^2}{4\pi}\left(\frac{r^2}{a^2} - \frac{a^2}{r^2}\right)\sin 2\theta\Sigma \sin\frac{3\pi}{2}\cos\frac{3\pi}{n}$$

$$\times \log\frac{1 - 2\frac{r}{a}\cos\left(\frac{3\pi}{2n} - \theta\right) + \frac{r^2}{a^2}}{1 + 2\frac{r}{a}\cos\left(\frac{3\pi}{2n} + \theta\right) + \frac{r^2}{a^2}}$$

$$\phi = \tfrac{1}{2}\omega r^2\frac{\sin 2\theta}{\cos\frac{\pi}{n}} + \frac{\omega a^2}{2\pi}\log\frac{1 + 2\frac{r^n}{a^n}\sin n\theta + \frac{r^{2n}}{a^{2n}}}{1 - 2\frac{r^n}{a^n}\sin n\theta + \frac{r^{2n}}{a^{2n}}}$$

$$- \frac{\omega a^2}{2\pi}\left(\frac{r^2}{a^2} - \frac{a^2}{r^2}\right)\cos 2\theta\Sigma \sin\frac{3\pi}{2}\sin\frac{3\pi}{n}$$

$$\times \tan^{-1}\frac{2\frac{r}{a}\sin\theta\left(\cos\frac{3\pi}{2n} - \frac{r}{a}\cos\theta\right)}{1 - 2\frac{r}{a}\cos\theta\cos\frac{3\pi}{2n} + \frac{r^2}{a^2}\cos 2\theta}$$

$$+ \frac{\omega a^2}{4\pi}\left(\frac{r^2}{a^2} + \frac{a^2}{r^2}\right)\sin 2\theta\Sigma \sin\frac{3\pi}{2}\sin\frac{3\pi}{n}$$

$$\times \log\left\{1 - 4\frac{r}{a}\cos\theta\cos\frac{3\pi}{2n} + 2\frac{r^2}{a^2}\left(\cos 2\theta - 2\cos^2\frac{3\pi}{2n}\right)\right.$$

$$-4\frac{r^3}{a^3}\cos\theta\cos\frac{s\pi}{2n}+\frac{r^4}{a^4}\Big\}$$

$$-\frac{\omega a^2}{4\pi}\left(\frac{r^2}{a^2}-\frac{a^2}{r^2}\right)\cos 2\theta\Sigma\sin\frac{s\pi}{2}\cos\frac{s\pi}{n}$$

$$\times\log\frac{1-2\frac{r}{a}\cos\left(\frac{s\pi}{2n}-\theta\right)+\frac{r^2}{a^2}}{1+2\frac{r}{a}\cos\left(\frac{s\pi}{2n}+\theta\right)+\frac{r^2}{a^2}}$$

$$-\frac{\omega a^2}{2\pi}\left(\frac{r^2}{a^2}-\frac{a^2}{r^2}\right)\sin 2\theta\Sigma\sin\frac{s\pi}{2}\cos\frac{s\pi}{n}$$

$$\times\tan^{-1}\frac{\sin\frac{s\pi}{2n}\left(\cos\frac{s\pi}{2n}-\frac{r}{a}\cos\theta\right)}{\cos\frac{s\pi}{n}-2\frac{r}{a}\cos\theta\cos\frac{s\pi}{2n}+\frac{r^2}{a^2}}.$$

ARRANGEMENTS OF m THINGS OF ONE SORT AND n THINGS OF ANOTHER SORT, UNDER CERTAIN CONDITIONS OF PRIORITY.

By *W. Allen Whitworth, M.A.*

OF the two integers m and n, let n (throughout the paper) be one which is not greater than the other.

The conditions of priority which I propose to consider are such as place limits on the proportion of the two classes of things in any part of the arrangement reckoned from its initial extremity.

For example, if an urn contain black balls and white balls which are to be drawn out in succession, the order in which they are drawn out may be limited by the condition that the number of white balls drawn must never exceed the number of black balls; or, again, by the condition that the excess of black balls over white must never be more than a given number.

I use the symbol $c_{r,\,n-r}$ in preference to c_r^n to denote the number of combinations of n things taken r at a time, and it is convenient to notice that

$$c_{m+1,\,n-1}=\frac{n}{m+1}\,c_{m,\,n};$$

and, therefore,

$$c_{m,n} - c_{m+1,n-1} = \frac{m-n+1}{m+1} c_{m,n} = \frac{m-n+1}{m+n+1} c_{m+1,n},$$

and, as a particular case,

$$c_{n,n} - c_{n+1,n-1} = \frac{1}{n+1} c_{n,n} = 2c_{n,n} - \tfrac{1}{2} c_{n+1,n+1}.$$

Also that $$c_{n,n-1} = \tfrac{1}{2} c_{n,n}.$$

1. From any origin O (fig. 13) draw OX horizontally and OY vertically, and let P be the point which would be reached by starting from O and taking m paces horizontally in the direction OX and n paces vertically parallel to OY.

Let A be the point reached by taking one pace horizontally and one pace vertically, B the point reached by taking two paces in each direction, C by taking three paces in each direction, and so on; so that $ABC\ldots$ lie in the straight line bisecting the right angle XOY.

Let o, a, b, $c\ldots$ be the points reached by taking one horizontal pace from each of the points O, A, B, $C\ldots$ and ω, α, β, $\gamma\ldots$ the points reached by taking one pace vertically from O, A, B, $C\ldots$ respectively.

In considering the number of routes from one point to another we shall proceed on the understanding that each route is to be traversed by paces taken only horizontally and vertically, and without retrogression.

2. The total number of routes from O to P must be the same as the number of different orders in which m horizontal paces and n vertical paces can be arranged. For in *any* order m horizontal paces and n vertical paces will take us from O to P and every different order will give a different route. Hence, the number of routes from O to P is

$$c_{m,n} \equiv \frac{\lfloor m+n}{\lfloor m \lfloor n} \equiv \frac{(m+n)(m+n-1)(m+n-2)\ldots(n+1)}{\lfloor m} \equiv \frac{(m+n)(m+n-1)(m+n-2)\ldots(m+1)}{\lfloor n}.$$

3. Let $f_{m,n}$ denote the number of routes from O to P which do not cross the diagonal line $OAB\ldots$, and let $x_{m,n}$ denote the number of routes which do cross this diagonal; so that

$$f_{m,n} + x_{m,n} = c_{m,n}.$$

4. All the routes included in the number $x_{m,n}$ must pass along one or other of the paces $O\omega$, $A\alpha$, $B\beta$...; some of the routes may pass along more than one, but we will classify all the routes according to the point at which they *first* cross the diagonal whether they recross more than once or not.

Let us consider the routes which cross the diagonal at K (after making say r horizontal and r vertical paces) without having crossed before, but independently of any consideration of the number of times they may cross afterwards.

Any such route $OK\kappa P$ may be divided into three portions OK, $K\kappa$, κP, of which

OK can be made in $f_{r,r}$ ways (Art. 3),

$K\kappa$ in one way,

κP in $c_{m-r,\,n-r-1}$ ways (Art. 2).

Hence, the whole number of ways in which the route can be made, or the number of routes through K, is

$$f_{r,r}\,c_{m-r,\,n-r-1}.$$

Giving r all values from 0 to $n-1$ we shall get all the routes comprehended in the number $x_{m,n}$.

Hence,

$$x_{m,n}=f_{0,0}c_{m,n-1}+f_{1,1}c_{m-1,n-2}+f_{2,2}c_{m-2,n-3}+\ldots+f_{n,n-1}c_{m-n+1,0},$$

or, in virtue of the relation,

$$f_{m,n}+x_{m,n}=c_{m,n},$$

$$f_{m,n}=c_{m,n}-f_{0,0}c_{m,n-1}-f_{1,1}c_{m-1,n-2}-f_{2,2}c_{m-2,n-3}-\ldots-f_{n-1,n-1}c_{m-n+1,0}.$$

5. Considering the particular case when $m=n$ we have

$$f_{n,n}=c_{n,n}-f_{0,0}c_{n,n-1}-f_{1,1}c_{n-1,n-2}-f_{2,2}c_{n-2,n-3}-\ldots-f_{n-1,n-1}c_{1,0},$$

or, in virtue of the identity

$$c_{r,r-1}=\tfrac{1}{2}c_{r,r},$$

$$f_{n,n}=c_{n,n}-\tfrac{1}{2}\{f_{0,0}c_{n,n}+f_{1,1}c_{n-1,n-1}+f_{2,2}c_{n-2,n-2}+\ldots+f_{n-1,n-1}c_{1,1}\}.$$

PROPOSITION I.

6. *To prove that*

$$f_{n,n}=\frac{c_{n,n}}{n+1}\equiv 2c_{n,n}-\tfrac{1}{2}c_{n+1,n+1}.$$

Observing that $c_{0,0}=1$, $c_{1,1}=2$, $c_{2,2}=6$, $c_{3,3}=20$, $c_{4,4}=70$, and actually counting the routes for the first five values of n,

$$f_{0,0}=1,\ f_{1,1}=1,\ f_{2,2}=2,\ f_{3,3}=5,\ f_{4,4}=14,$$

we see that the proposition is true as long as n does not exceed 4.

To prove it generally, we shall show that if it be true when $n=0, 1, 2, \ldots$ or $x-1$, it will also be true when $n=x$.

By Art. 5 we have

$$f_{x,x}=c_{x,x}-\tfrac{1}{2}\{f_{0,0}c_{x,x}+f_{1,1}c_{x-1,x-1}+f_{2,2}c_{x-2,x-2}+\ldots+f_{x-1,x-1}c_{1,1}\}.$$

But, since the theorem is by hypothesis true when $n=0$, $1, 2, \ldots$ or $x-1$, we have

$$f_{0,0}=2c_{0,0}-\tfrac{1}{2}c_{1,1},$$
$$f_{1,1}=2c_{1,1}-\tfrac{1}{2}c_{2,2},$$
$$\&c.=\&c.,$$
$$f_{x-1,x-1}=2c_{x-1,x-1}-\tfrac{1}{2}c_{x,x}.$$

Hence, substituting, we get

$$f_{x,x}=c_{x,x}-\{c_{0,0}c_{x,x}+c_{1,1}c_{x-1,x-1}+c_{2,2}c_{x-2,x-2}+\ldots+c_{x-1,x-1}c_{1,1}\}$$
$$+\tfrac{1}{2}\{c_{1,1}c_{x,x}+c_{2,2}c_{x-1,x-1}+c_{3,3}c_{x-2,x-2}+\ldots+c_{x,x}c_{1,1}\}.$$

But we know that

$$c_{0,0}c_{x,x}+c_{1,1}c_{x-1,x-1}+\ldots+c_{x,x}c_{0,0}\equiv 2^{2x}$$

(or we may obtain it by expanding $(1-4x)^{-\frac{1}{2}}$ by Bin. Theor.; squaring, and equating coeffs. of x^n), therefore

$$f_{x,x}=c_{x,x}-\{2^{2x}-c_{x,x}c_{0,0}\}+\tfrac{1}{2}\{2^{2x+2}-2c_{x+1,x+1}c_{0,0}\},$$

or

$$f_{x,x}=2c_{x,x}-\tfrac{1}{2}c_{x+1,x+1},$$

which shows that if the theorem is true for the first x values of n, it is true for the next value.

But we have shown that it was true for initial values, therefore, universally,

$$f_{n,n}=2c_{n,n}-\tfrac{1}{2}c_{n+1,n+1},$$

or (which is the same thing),

$$f_{n,n}=\frac{1}{n+1}\,c_{n,n},$$

or again (which is the same thing),

$$f_{n,n}=c_{n,n}-c_{n+1,n-1}.$$

7. Since all the routes from O to P commence at the point O on the diagonal and terminate at the point P to the right of the diagonal, they may be classified according to the points at which they *first* pass to the right of the diagonal: *i.e.* according as they first pass along Oo or Aa or Bb, &c. Consider those which pass along Kk, any such route consists of three portions OK, kK, kP, of which OK lies altogether to the right of the diagonal, and can therefore be described in $\frac{1}{r+1}c_{r,r}$ ways; Kk can be described in one way, and kP can be described in $c_{m-r-1,\,n-r}$ ways. Hence there are

$$\frac{1}{r+1}c_{r,r}c_{m-r-1,\,n-r}$$

such routes, and giving r all values from 0 to n inclusive we must obtain all the $c_{m,n}$ routes from O to P. Therefore we must have identically

$$c_{m,n} \equiv c_{0,0}c_{m-1,n} + \tfrac{1}{2}c_{1,1}c_{m-2,n-1} + \tfrac{1}{3}c_{2,2}c_{m-3,n-2} + \cdots + \frac{1}{n+1}c_{n,n}c_{m-n-1,0}.$$

This is true for all values of m and n; therefore, writing $m+1$ and $n-1$ for m and n respectively, we obtain

$$c_{m+1,n-1} \equiv c_{0,0}c_{m,n-1} + \tfrac{1}{2}c_{1,1}c_{m-1,n-2} + \tfrac{1}{3}c_{2,2}c_{m-2,n-3} + \cdots + \frac{1}{n}c_{m,n-1}c_{m-n+1,0}.$$

PROPOSITION II.

8. *To prove that* $f_{m,n} = c_{m,n} - c_{m+1,n-1}$.

From Art. 4 we have

$$f_{m,n} = c_{m,n} - f_{0,0}c_{m,n-1} - f_{1,1}c_{m-1,n-2} - f_{2,2}c_{m-2,n-3} - \cdots - f_{n-1,n-1}c_{m-n+1,0}.$$

Substituting for $f_{0,0}$, $f_{1,1}$, $f_{2,2}$ their values as found in Art 6, in the form

$$f_{n,n} = \frac{1}{n+1}c_{n,n},$$

we obtain

$$f_{m,n} = c_{m,n} - c_{0,0}c_{m,n-1} - \tfrac{1}{2}c_{1,1}c_{m-1,n-2} - \tfrac{1}{3}c_{2,2}c_{m-2,n-3} - \cdots - \frac{1}{n}c_{n-1,n-1}c_{m-n+1,0},$$

or in virtue of the identity proved in Art. 7

$$f_{m,n} = c_{m,n} - c_{m+1,n-1}. \qquad \text{Q. E. D.}$$

N.B. We may also write the result in the form

$$f_{m,n} = \frac{m-n+1}{m+n+1} c_{m+1,n} \text{ or } f_{m,n} = \frac{m-n+1}{m+1} c_{m',n}.$$

9. If, in Art. 7, the routes instead of starting from O had started from the point J situated h paces horizontally to the left of O, the whole number of routes would have been $c_{m+h',n}$. And classifying them as in the case of Art. 7, any route passing along Kk would have consisted of three parts, JK, Kk, kP, of which

JK could be described in $\frac{h+1}{r+h+1} c_{r+h,r}$ ways (Art. 8),

Kk in one way,

kP in $c_{m-r-1,n-r}$ ways.

Hence there are $\frac{h+1}{r+h+1} c_{m-r-1,n-r} c_{r+h,r}$ such routes. And giving r all values from 0 to n we obtain the identity

$$c_{m+h',n} \equiv c_{h,0} c_{m-1',n} + \frac{h+1}{h+2} c_{h+1,1} c_{m-2,n-1} + \frac{h+1}{h+3} c_{h+2,2} c_{m-3,n-2} + \ldots + \frac{h+1}{h+n+1} c_{h+n,h} c_{m-n-1',0}$$

or

$$\frac{c_{m+h,n}}{h+1} \equiv \frac{c_{h,0} c_{m-1,n}}{h+1} + \frac{c_{h+1,1} c_{m-2,n-1}}{h+2} + \frac{c_{h+2,2} c_{m-3,n-2}}{h+3} + \ldots + \frac{c_{h+n,n} c_{m-n-1,0}}{h+n+1}.$$

Write h for $h+1$ and m for $m-1$, and we have

$$\frac{c_{m+h,n}}{h} \equiv \frac{c_{h-1,0} c_{m,n}}{h} + \frac{c_{h,1} c_{m-1,n-1}}{h+1} + \ldots + \frac{c_{h+n-1,n} c_{m-n,0}}{h+n}.$$

Now write $n-h$ for n, and we have

$$\frac{c_{m+h',n-h}}{h} \equiv \frac{c_{h-1',0} c_{m,n-h}}{h} + \frac{c_{h,1} c_{m-1,n-h-1}}{h+1} + \ldots + \frac{c_{n-1,n-h} c_{m-n+h',0}}{n}.$$

PROPOSITION III.

10. *To find the number of routes from O to P which touch or cross a line HR parallel to the diagonal OK at a vertical distance of h paces above it.*

The routes may be classified according to the points at which they first touch the line HR.

Let $O\lambda' LP$ be any route which first reaches this line at a point L, distant r horizontal paces and $r+h$ vertical paces from O, and let $\lambda' L$ be the pace by which the point L is approached.

The route may be divided into three parts $O\lambda'$, $\lambda' L$, and LP; of which

$O\lambda'$ can be made in $\frac{h}{r+h} c_{r+h-1,\, r}$ ways (Art. 7),

$\lambda' L$ in one way,

LP in $c_{m-r,\, n-r-h}$ ways;

therefore the number of routes first touching at L is

$$\frac{h}{r+h} c_{r+h-1,\, r} c_{m-r,\, n-r-h},$$

and the whole number of routes will be got by giving r all values from 0 to $n-h$ inclusive, and adding.

The summation is that of the final series of Art. 9.

Hence, the whole number of routes required is $c_{m+h,\, n-h}$.

11. COROLLARY. It follows that *the number of routes from O to P which do not cross or touch the diagonal h paces above the diagonal through O, is*

$$c_{m,\, n} - c_{m+h,\, n-h},$$

and writing $h+1$ for h, *the number of routes from O to P which do not cross* (but may touch) *the diagonal h paces above the diagonal through O, is*

$$c_{m,\, n} - c_{m+h+1,\, n-h-1},$$

for if they do not *cross* this diagonal, they cannot *touch* the diagonal next beyond it.

Observe that if $h=0$ this reduces to the case of Prop. II. (Art. 8).

12. EXAMPLES.

(1) *A man drinks in random order n glasses of wine and n glasses of water (all equal); shew that the odds are n to 1 against his never having drunk throughout the process more wine than water* (*Educational Times*, June, 1878, question 5669).

Let glasses of water be represented by horizontal paces and the glasses of wine by vertical paces, then the chance required must be the same as the chance of keeping always to the right of the diagonal OV in passing at random from O to V.

The total number of routes (by Art. 2) is $c_{n,n}$.

The number which keep to the right of the diagonal (by Art. 6) is $c_{n,n} \div (n+1)$.

Therefore the chance is $1 \div (n+1)$, or the odds are n to 1. Q. E. D.

(2) *If n men and their wives go over a bridge in single file, in random order, subject only to the condition that there are to be never more men than women gone over, prove that the chance that no man goes over before his wife is* $(n+1)\,2^{-n}$ (*Educational Times*, September 1878, Question 5744).

Regarding the different men as indifferent and the different women as indifferent, the number of orders in which they could go over would be (as in Art. 2) $c_{n,n}$ and the number of orders subject to the condition that never more men than women are gone over would be (as in Art. 6) $c_{n,n} \div (n+1)$.

But the men may be arranged among themselves in $\lfloor\underline{n}$ ways, and the women in $\lfloor\underline{n}$ ways.

Therefore the total number of possible orders, subject to the given condition, is

$$\lfloor\underline{n}\,\lfloor\underline{n}\, c_{n,n} \div (n+1) \text{ or } \lfloor\underline{2n} \div (n+1).$$

These include amongst them all the orders in which each man has gone over before his own wife. But amongst all the possible $\lfloor\underline{2n}$ orders in which the $2n$ persons could cross, any assigned man will be before his wife in $(\frac{1}{2})\lfloor\underline{2n}$ orders and all the n men will be before their wives in $(\frac{1}{2})^n\lfloor\underline{2n}$ orders.

Therefore the required chance is

$$\frac{(\frac{1}{2})^n\lfloor\underline{2n}}{\lfloor\underline{2n} \div (n+1)} \text{ or } (n+1)\,2^{-n}. \qquad \text{Q. E. D.}$$

(3) *In how many orders can m positive units and n negative units be arranged so that the sum to any number of terms may never be negative?*

By taking horizontal paces to represent positive units and vertical paces to represent negative units, this question is

seen to be equivalent to that of Art. 3. And, therefore, (by Art. 8) we have the result

$$c_{m,n} - c_{m+1,n-1} = (m-n+1)\frac{\lfloor m+n}{\lfloor m+1 \lfloor n}.$$

(4) *In how many orders can m even powers of x and n odd powers of x be arranged, so that when* $x=-1$ *the sum to any number of terms may never be negative?*

This question differs from the preceding in that the terms are all different. The positive terms may therefore be permuted amongst themselves in $\lfloor m$ orders and the negative terms in $\lfloor n$ orders. The required number of ways is therefore

$$\frac{(m-n+1)\lfloor m+n}{m+1}.$$

(5) *In how many orders can a man win m games and lose n games so as at no period to have lost more than he has won?*

The question is equivalent to question (3) and the answer is identically the same.

(6) *A man possessed of* $a+1$ *pounds plays even wagers for a stake of* 1 *pound. Find the chance that he is ruined at the* $(a+2x+1)^{\text{th}}$ *wager and not before.*

He necessarily loses the last wager.

Let $m=a+x$, the number of wagers he loses before the last one.

$n=x$, the number he wins.

The number of orders in which his gains and losses can be arranged will be represented by the number of routes from P to O (fig. 13) without crossing the diagonal OV. And this must be the same as the number of routes from O to P which is {by Art. 8)

$$\frac{m-n+1}{m+1}c_{m,n};$$

and, therefore, the chance that m loses and n gains are arranged in such order that the man is not ruined before the $(m+n)^{\text{th}}$ wager is

$$\frac{m-n+1}{m+1}.$$

But the chance that his first m wagers should give m losses and n gains is

$$\frac{\lfloor m+n}{\lfloor m \lfloor n} \left(\tfrac{1}{2}\right)^{m+n},$$

and the chance that the final wager gives a loss is $\frac{1}{2}$.

Therefore the chance that he is not ruined before the $(a+2x+1)^{\text{th}}$ wager, and that he is ruined then, is

$$\frac{m-n+1}{m+1} \frac{\lfloor m+n}{\lfloor m \lfloor n} \left(\tfrac{1}{2}\right)^{m+n+1} = \frac{(a+1) \lfloor a+2x}{\lfloor a+x+1 \lfloor x} \left(\tfrac{1}{2}\right)^{a+2x+1}.$$

(7) *In how many different orders can a man possessed of h pounds win m wagers and lose n wagers of 1 pound each without being ruined during the process?*

The number of ways must be the same as the number of routes from O to P without touching the diagonal JR. The result is therefore (by Art. 11)

$$c_{m,\,n} - c_{m+h,\,n-h}.$$

(8) *If a man playing for a constant stake, win $2n$ games and lose n games, the chance that he is never worse off than at the beginning and never better off than at the end is*

$$\frac{n^2+n+2}{4n^2+6n+2}.$$

(9) *If he win $2n+1$ games and lose $n+1$ games, the chance is*

$$\frac{n}{4n+6}.$$

(*Educational Times*, November, 1878, Question 5804).

ON CENTRES OF PRESSURE, METACENTRES, &c.

By *T. C. Lewis, M.A.*, Fellow of Trinity College, Cambridge.

THE centre of pressure of a plane area bounded by a closed conic may be found by the following elementary geometrical method.

(i) Through every point of the boundary of the given area draw vertical lines to the surface; then if the vertical

line through the centre of gravity of the liquid enclosed by these lines meet the plane area in H, H is the centre of pressure required; for the centre of parallel forces is not altered if the forces are altered in any constant ratio, or if their directions are changed, provided only their points of application and their parallelism are preserved. The problem of finding the centre of pressure for an area is therefore identical with that of determining the vertical line through the centre of gravity of the liquid which lies vertically over it.

(ii) Let ABC be any closed conic section touching the surface at A (fig. 14); let C be the other point of the conic at which the tangent is parallel to the surface, *i.e.* the other end of the diameter through A. Through C draw a vertical line, and take in it some point O above the surface of the liquid. Construct the cone whose vertex is O and base ABC, let the section made by the surface of the liquid be Abc, c being on OC. Bisect AC, Ac in D, E, respectively; join DO, EO; take F, G in DO, EO respectively, at one quarter of the distance from D, E towards O; join GF, and produce to intersect the given area ABC in H; H will be the required centre of pressure.

For Abc is a conic section, and the tangent at c is parallel to that at C, *i.e.* to that at A; therefore D, E are the centres of the two sections; therefore F, G are the centres of gravity of the two cones $OABC$, $OAbc$; therefore the centre of gravity of the liquid between the two sections must lie in GF. Also GF is parallel to DE, and DE is parallel to Cc, which is vertical; therefore GF is vertical, and is in the plane OCA, therefore H is on AC, and

$$DH : DC :: DF : DO :: 1 : 4.$$

These two results, that GF is vertical, and that DH is one quarter of DC, are independent of the position of O on the vertical through C, and of the angle which the plane ABC makes with the surface. They are therefore true whatever this angle is, and whatever the distance CO is. Take O at infinity; the wedge of liquid between the two sections then becomes the same as the liquid vertically over the plane area ABC; and H, being determined as before, is seen to be the centre of pressure required.

(iii) If a circle be immersed so that its highest point is at a depth h below the surface, its centre, O, at a depth $h+k$, and if a be its radius, and $OP = \frac{1}{4}a$, measured along the radius

to the lowest point, the centre of pressure is at Q in OP (fig. 15), such that

$$OQ : QP :: k : h.$$

A similar construction holds for an ellipse.

Radius of gyration.

The radius of gyration, k, of a plane figure, of area A, about any axis of y lying in its plane, may be defined by the equation

$$k^2 A = \Sigma x^2 d\sigma,$$

$d\sigma$ being any small element of the area, and Σ denoting a summation extending to all such elements.

Some useful results may be obtained without the aid of Integral Calculus.

(i) Let the area be immersed in a liquid so as to touch the surface; let the tangent in the surface be taken as axis of y: let $\bar{x}_1$, $\bar{x}_2$ be the distances of the centre of gravity and centre of pressure of the area respectively from the axis of y. The pressure at a point of the area at a distance x from the axis of y is proportional to x, call it θx; therefore $\bar{x}_1$ and $\bar{x}_2$ are determined by the equations

$$\bar{x}_2 \Sigma \theta x d\sigma = \Sigma \theta x^2 d\sigma = \theta k^2 A,$$

and

$$\bar{x}_1 A = \Sigma x d\sigma;$$

therefore, by multiplying these two equations together, we get

$$\bar{x}_2 \bar{x}_1 = k^2;$$

or, the radius of gyration of the plane area, just immersed, about the tangent in the surface of the liquid is a mean proportional between the distances of the centre of gravity and centre of pressure of the area from that tangent.

(ii) Let

$$x = \bar{x}_1 + x_1,$$

and let k_0 be the radius of gyration about an axis through the centre of gravity of the area, and parallel to the axis of y, then

$$k^2 A = \Sigma x^2 d\sigma = \Sigma (\bar{x}_1 + x_1)^2 d\sigma$$

$$= x_1^2 A + \Sigma x_1^2 d\sigma = \bar{x}_1^2 A + k_0^2 A$$

therefore

$$k^2 = k^2 - \bar{x}_1^2.$$

(iii) Applying this to the case of a triangle having one point, A, in the surface, and the side BC horizontal at a distance p_1 from A, we have

$$\bar{x}_1 = \tfrac{2}{3}p_1 \text{ and } \bar{x}_2 = \tfrac{3}{4}p_1;$$

therefore $$k^2 = \tfrac{1}{2}p_1^2;$$

therefore $$k_0^2 = (\tfrac{1}{2} - \tfrac{4}{9})p_1^2 = \tfrac{1}{18}p_1^2.$$

(iv) If BC be in the surface,

$$\bar{x}_1 = \tfrac{1}{3}p_1 \text{ and } \bar{x}_2 = \tfrac{1}{2}p_1;$$

therefore $$k^2 = \tfrac{1}{6}p_1^2,$$

and $$k_0^2 = (\tfrac{1}{6} - \tfrac{1}{9})p_1^2 = \tfrac{1}{18}p_1^2, \text{ as before.}$$

(v) If we take a parallelogram with one side in the surface, and the parallel side at a distance $2a$ from it, we have

$$\bar{x}_1 = a, \ \bar{x}_2 = \tfrac{4}{3}a;$$

therefore $$k^2 = \tfrac{4}{3}a^2,$$

and $$k_0^2 = \frac{a^2}{3}.$$

(vi) If we consider the case of a closed conic touching the surface in a line whose perpendicular distance from the centre is p, we have, as shewn above,

$$\bar{x}_1 = p, \ \bar{x}_2 = \tfrac{5}{4}p.$$

therefore $$k^2 = \tfrac{5}{4}p^2;$$

therefore $$k_0^2 = \tfrac{1}{4}p^2.$$

Metacentres.

All the radii of gyration found in the previous note are useful for finding metacentres.

(i) If V be the volume of liquid displaced by a floating solid, A the area of the section made by the plane of floatation, k_0 the radius of gyration of A about an axis through its centre of gravity and perpendicular to a plane which divides the solid symmetrically, H the centre of gravity of the displaced liquid, and M the metacentre, then it may be shown in an elementary manner (vid. Besant's *Hydromechanics*, 3rd edition, Art. 62), that

$$HM = \frac{Ak_0^2}{V}.$$

Let the immersed part of the solid be an inverted cone with its vertex, O, vertically below the centre of gravity of the area A, and at a depth h, then $V=\frac{1}{3}Ah$; therefore

$$HM=\frac{3k_0^2}{h},$$

and

$$OM=\tfrac{3}{4}h+\frac{3k_0^2}{h}=\tfrac{3}{4}\frac{h^2+4k_0^2}{h}.$$

(ii) If A be an isosceles triangle, and p_1 the perpendicular from the vertex on the base,

$$OM=\tfrac{3}{4}\frac{h^2+\frac{2}{3}p_1^2}{h},$$

by (iii) of preceding note.

If A be an equilateral triangle, and OA, OB, OC make an angle α with the vertical, $\frac{2}{3}p_1=h\tan\alpha$; therefore

$$OM=\tfrac{3}{4}h\,(1+\tfrac{1}{2}\tan^2\alpha).$$

(iii) If A be a rectangle with sides $2a$, $2b$,

$$OM=\tfrac{3}{4}\frac{h^2+\frac{4}{3}a^2}{h} \text{ or } \tfrac{3}{4}\frac{h^2+\frac{4}{3}b^2}{h},$$

for the two symmetrical displacements.

If A be a square, and the lines joining O to its angular points make an angle α with the vertical, $\sqrt{(2)}\,a=h\tan\alpha$; therefore

$$OM=\tfrac{3}{4}h\,(1+\tfrac{2}{3}\tan^2\alpha).$$

(iv) If A be an ellipse with semi-axes a and b,

$$OM=\tfrac{3}{4}\frac{h^2+a^2}{h} \text{ or } \tfrac{3}{4}\frac{h^2+b^2}{h},$$

for the two symmetrical displacements.

If A be a circle, and the semi-vertical angle of the inverted cone be α,

$$a=h\tan\alpha;$$

therefore

$$OM=\tfrac{3}{4}h\,(1+\tan^2\alpha)$$
$$=\tfrac{3}{4}h\sec^2\alpha.$$

September 2, 1878.

CAUCHY'S THEOREM REGARDING THE DIVISIBILITY OF $(x+y)^n+(-x)^n+(-y)^n$.

By *Thomas Muir, M.A., F.R.S.E.*

CAUCHY'S theorem in regard to the divisibility of $(x+y)^{2n+1}-x^{2n+1}-y^{2n+1}$ to which Mr. Glaisher has recently drawn attention may be established in a simple manner quite distinct from that employed by its author.

Let $$S_n=(x+y)^n+(-x)^n+(-y)^n,$$

then, by ordinary multiplication, we have

$$\begin{aligned}
2S_3S_n+3S_2S_{n+1} = \\
&2(x+y)^{n+3} &&+2(x+y)^3(-x)^n &&+2(x+y)^3(-y)^n \\
&+2(-x)^3(x+y)^n &&+2(-x)^{n+3} &&+2(-x)^3(-y)^n \\
&+2(-y)^3(x+y)^n &&+2(-y)^3(-x)^n &&+2(-y)^{n+3} \\
&+3(x+y)^{n+3} &&+3(x+y)^2(-x)^{n+1} &&+3(x+y)^2(-y)^{n+1} \\
&+3(-x)^2(x+y)^{n+1} &&+3(-x)^{n+3} &&+3(-x)^2(-y)^{n+1} \\
&+3(-y)^2(x+y)^{n+1} &&+3(-y)^2(-x)^{n+1} &&+3(-y)^{n+3}.
\end{aligned}$$

Now the sum of the 1st, 5th, 9th, 10th, 14th, 18th terms here clearly

$$=5S_{n+3};$$

the sum of the 4th, 7th, 13th, 16th

$$\begin{aligned}
&=-2(x+y)^n(x^3+y^3)+3(x+y)^{n+1}(x^2+y^2) \\
&=(x+y)^{n+1}\{-2x^2+2xy-2y^2+3x^2+3y^2\} \\
&=(x+y)^{n+3};
\end{aligned}$$

the sum of the 2nd, 8th, 11th, 17th

$$\begin{aligned}
&=2(-x)^n\{(x+y)^3-y^3\}+3(-x)^{n+1}\{(x+y)^2+y^2\} \\
&=(-x)^{n+1}(-2x^2-6xy-6y^2+3x^2+6xy+6y^2) \\
&=(-x)^{n+3};
\end{aligned}$$

and, similarly, the sum of the 3rd, 6th, 12th, 15th

$$=(-y)^{n+3}.$$

Consequently we have the identity

$$\left.\begin{aligned} 2S_3S_n + 3S_2S_{n+1} &= 6S_{n+2} \\ (\tfrac{1}{3}S_3)\,S_n + (\tfrac{1}{2}S_2)\,S_{n+1} &= S_{n+2} \end{aligned}\right\} \text{ or } \quad \ldots\ldots\ldots (A).$$

In this, putting $n = 1, 2, 3$, and noting that $S_1 = 0$, we see that

S_4 is divisible by $(\frac{1}{2}S_2)^2$,

S_5 is divisible by $\frac{1}{2}S_2$ and $\frac{1}{3}S_3$,

S_6 is divisible by neither.

Then, as S_7 is got by multiplying S_5 by $\frac{1}{2}S_2$ and S_4 by $\frac{1}{3}S_3$ and adding, it follows that

S_7 is divisible by $(\frac{1}{2}S_2)^2$ and $\frac{1}{3}S_3$;

as S_8 is got in the same way from S_6 and S_5 it follows that

S_8 like S_2 is divisible by $\frac{1}{2}S_2$;

and, as S_9 is got in the same way from S_7 and S_6, it follows that

S_9 like S_3 is divisible by $\frac{1}{3}S_3$.

Consequently

S_{10} like S_4 is divisible by $(\frac{1}{2}S_2)^2$,

S_{11} like S_5 is divisible by $\frac{1}{2}S_2$ and $\frac{1}{3}S_3$,

and so on in a cycle of six elements.

Now $$\tfrac{1}{2}S_2 = x^2 + xy + y^2,$$

and $$\tfrac{1}{3}S_3 = xy\,(x+y);$$

hence we have the following theorem which includes Cauchy's

$$\left.\begin{aligned} &\textit{The expression } (x+y)^n + (-x)^n + (-y)^n \textit{ is divisible} \\ &\textit{by neither } xy\,(x+y) \textit{ nor } x^2+xy+y^2 \textit{ when } n = 6m \\ &\textit{by } xy\,(x+y) \textit{ and } (x^2+xy+y)^2 \textit{ when } n = 6m+1 \\ &\textit{by } x^2+xy+y^2 \textit{ when } n = 6m+2 \\ &\textit{by } xy\,(x+y) \textit{ when } n = 6m+3 \\ &\textit{by } (x^2+xy+y^2)^2 \textit{ when } n = 6m+4 \\ &\textit{and by } xy\,(x+y) \textit{ and } x^2+xy+y^2 \textit{ when } n = 6m+5 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (B).$$

NOTE ON CAUCHY'S THEOREM.

By *J. W. L. Glaisher.*

THE resolutions of $(x+y)^n - x^n - y^n$ for $n = 9$, 11, and 13, which I gave on p 53*, were, I have found, given by Cauchy himself in his "Note sur quelques théorèmes d'algèbre" (*Exercises d'analyse et de physique mathématique*, t. ii, 1841, pp. 137–144).

The general formula for $(x+y)^n + (-x)^n + (-y)^n$ in terms of $xy(x+y)$ and x^2+xy+y^2 is given by Todhunter in the second edition of his *Theory of Equations* (1867), p. 189. The subject is also considered in two papers by Mr. Muir and myself in the current number of the *Quarterly Journal of Mathematics*.†

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, November 14, Lord Rayleigh, F.R.S., *President*, in the Chair. After the Treasurer's and Secretaries' Reports had been read and adopted, the meeting proceeded to the election of the new Council. The Scrutators (Messrs. C. Pendlebury and R. F. Scott) declared the following gentlemen elected:

President, Mr. C. W. Merrifield, F.R.S.; *Vice-Presidents*, Prof. Cayley, F.R.S. and Lord Rayleigh, F.R.S.; *Treasurer*, Mr. S. Roberts, F.R.S.; *Hon. Secs.*, Messrs. M. Jenkins and R. Tucker; *Other Members*, Mr. J. W. L. Glaisher, F.R.S.; Mr. H. Hart; Dr. Henrici, F.R.S.; Dr. Hirst, F.R.S.; Dr. Hopkinson, F.R.S.; Mr. A. B. Kempe; Dr. Spottiswoode, F.R.S.; Prof. H. J. S. Smith, F.R.S.; Mr. H. M. Taylor; and Mr. J. J. Walker.

Mr. Merrifield having taken the chair, Mr. J. D. H. Dickson, M.A., was elected a member, and the Rev. A. Freeman and Prof. Reinold were admitted into the Society. Prof. W. S. Jevons, F.R.S. was proposed for election. The Chairman read and laid before the meeting a letter addressed to the Society by Mr. Warren de la Rue, F.R.S., respecting a memorial to M. Leverrier.

The following communications were made to the Society: "On the instability of Jets," Lord Rayleigh; "On self-strained Frames of Six Joints," Prof. Crofton, F.R.S. (read by Mr. Hart, the two conclusions established were (i); in order that a hexagonal frame *ABCDEF*, stiffened by means of diagonal bars *AD*, *BE*, *CF*, shall be capable of being selfstrained, it is necessary and sufficient that *the six joints lie on a conic*, (ii) there is only one other case where a frame of 6 joints and 9 bars can be self-strained, *i.e.* when the lines *BA*, *DE*, *CF* meet in one point); "On the Calculus of Equivalent Statements, H. McColl, B.A., (an abstract, read by Mr. Tucker, stated that the paper contained the solution of a test problem to show the power of the author's method of elimination; then an explanation with illustrations and applications of another allied method, called "The method of unit and zero substitution"; next, a brief indication of the way in which the algebra of logic may render important service to scientific men in investigating the causes of natural phenomena; and, lastly, a brief criticism of Prof. Jevons's method of solving logical problems).

R. TUCKER, M.A. *Hon. Sec.*

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* *On a class of algebraical relations*, § 1.

† *On an expansion of* $(x+y)^n + (-x)^n + (-y)^n$ *Quarterly Journal*, vol. XVI. p. 9, *On Cauchy's theorem relating to the factors of* $(x+y)^n - x^n - y^n$, *Ibid.* p. 89.

MATHEMATICAL NOTES.

Geometrical demonstration of a known Theorem relating to Surfaces.

TAKE an infinitely small arc aa_1 (figs. 16 and 17) of a curve traced on a surface. Draw at the extremities of this arc the normals an_1, a_1n_1 to the surface. The angle which a_1n_1 makes with the plane naa_1 is the *angle of geodesic torsion.*

Suppose that the curve (a) is the intersection of two surfaces cutting one another at the same angle along this curve. By drawing at a the normals an, am to each of the surfaces, and also the normals a_1n_1, a_1m_1 at a_1 we have the angles man, $m_1a_1n_1$ which are equal to one another.

We can bring these angles into coincidence by means of two rotations; one, which brings into coincidence the planes of these angles, takes place round the characteristic of the plane *man*; the other takes place round a perpendicular to this plane, drawn from its focus. The characteristic of the plane *man* is the axis of curvature C of the curve (a); the perpendicular to the plane *man* drawn from the focus of this plane is the tangent aa_1 to this curve.

Turning round C, a becomes a_1, and the angle *man* becomes ma_1n; then this angle turns round a_1 and becomes coincident with $m_1a_1n_1$.

It follows that the angle m_1a_1m is equal to n_1a_1n, and as aa_1 is normal to the plane of the moving angle, these angles measure the angles which the normals a_1m_1, a_1n_1 make with the planes maa_1, naa_1, viz. these are the angles of geodesic torsion of a relative to the surfaces of which this curve is the intersection.

We thus obtain a geometrical demonstration of the known theorem: *The angles of geodesic torsion of the curve of intersection of two surfaces which cut one another at a constant angle are equal.*

A. MANNHEIM.

On Triangles self-conjugate with respect to a Parabola.

Observe the following construction: The diameter intercepted between a point and its polar is bisected by the parabola in a point at which the tangent is parallel to the polar. In other words, the tangent to a parabola parallel to a given straight line lies midway between that line and

its pole the point of contact and the pole being on the same diameter.

Now let ABC (fig. 18) be a triangle self-conjugate with respect to a parabola; D, E, F the middle points of BC, CA, AB; P, Q, R the points where the diameters through A, B, C cut the curve; then the above construction shows that the sides of the triangle DEF will be the tangents at P, Q, R.* Also since the tangents at Q, R intersect at D a point on BC the polar of A, the chord QR will pass through A.

II. The perpendicular from A upon its polar being parallel to the normal at P, the portions of both intercepted by the axis, and their projections upon that axis will be equal. This is an extension of the property that the subnormal is constant.

III. The area of the triangle ABC will be double of that of PQR. Let Q lie between P and R and produce BQ to meet AC in Y. Then since BY is parallel to PA, and $BQ = QY$, $BQP = BQA = \frac{1}{2}BYA$. Similarly $BQR = BQC = \frac{1}{2}BYC$. Hence $PQR = \frac{1}{2}ABC$.

IV. The orthocentre of the triangle DEF is the centre of the circle circumscribing ABC, and the circle circumscribing the former is the nine-point circle of the latter. The directrix passes through the one, the focus is situate on the other. An independent proof may be subjoined of the former of these propositions.

Let the diameters AP, CR meet the directrix in L, N, respectively. The angle LEN is double of DEF, the triangle LEN is isosceles and therefore bisected by the perpendicular from E upon the directrix. (This is the triangle used in Drew's Geometrical Conics for drawing a pair of tangents). Compare this with the triangle formed by joining A, C to O the centre of the circle circumscribing ABC. The vertical angle of this isosceles triangle AOC is double of ABC, *i.e.* of DEF, and therefore the triangle is similar to LEN. And the base LN is the projection of AC upon the directrix, through an angle equal to that between OE which is perpendicular to AC and the perpendicular from E upon the directrix. This perpendicular must therefore equal the

* I find that this theorem is given in Wolstenholme's *Problems*, No. 442 (ed. 1, 1867). It was set by me in an examination paper in 1862. Steiner's theorem that the orthocentre of DEF (§ IV) lies on the directrix was originally proved with the help of Pascal's hexagram. For another elementary proof, see the *Lady's and Gentlemen's Diary*, 1868.

projection of OE through the angle between them, therefore O must be a point on the directrix.

The following properties are therefore established: viz.

(1) *If a triangle be self-conjugate with respect to a parabola the straight lines joining the middle points of its sides are tangents to the parabola; and touch the curve in points the diameter through one of which meets the chord joining the other two in one of the angular points of the self-conjugate triangle.*

(2) *If a perpendicular be drawn from any point upon its polar with respect to a parabola, the difference of the abscissæ of the point where it cuts the axis and of the given point from which it was drawn is constant.*

(3) *The area of a self-conjugate triangle is double of that formed by joining the points of contact of the three tangents parallel to its sides.*

(4) *The centre of the circle circumscribing a self-conjugate triangle lies on the directrix of the parabola.*

(5) *The nine-point circle of such a triangle passes through the focus.*

A. F. Torry.

Note on the Centre of Gravity of the Frustrum of a Pyramid.

Let $ABCabc$ (fig. 19) be the frustrum; ABC, abc its parallel faces; F, f their centres of gravity, and G that of the frustrum.

Divide the frustrum into three pyramids by the planes Abc, Bbc. Then clearly

$$\frac{\text{vol. of } cABC}{a^3} = \frac{\text{vol. of } Aabc}{b^3} = \frac{\text{vol. of } ABbc}{ab},$$

where $$a = AB, \quad b = ab.$$

Now the centre of gravity of a pyramid is that of four equal particles placed at the vertices; hence, we may replace the three pyramids first formed by three sets of four particles proportional to the volumes of the pyramids, and so replace the frustrum by the following system of particles

a^3, $a^3 + ab$, $a^3 + ab + b^3$, at C, B, A, respectively;

b^3, $b^3 + ab$, $b^3 + ab + a^3$, at a, b, c, respectively.

But two other similar arrangements can be similarly obtained. Superimposing these three systems, we find that the centre of gravity of the frustrum must be that of

three equal particles at A, B, C, proportional to $3a^2+2ab+b^2$,

and „ „ a, b, c, „ $3b^2+2ab+a^2$,

hence, we have at once

$$\frac{FG}{3b^2+2ab+a^2}=\frac{fG}{3a^2+2ab+b^2}=\frac{Ff}{4(a^2+ab+b^2)}.$$

J. W. SHARPE.

A Formula by Gauss for the Calculation of log 2 and certain other Logarithms.

Gauss has given, Werke, t. II., p. 501, a formula which is in effect as follows:

$$2^{196}=10^{59}\left(\frac{1025}{1024}\right)^5\left(\frac{1048576}{1048575}\right)^8\left(\frac{6560}{6561}\right)^3\left(\frac{15624}{15625}\right)^8\left(\frac{9801}{9800}\right)^4,$$

viz., this is

$$=2^{59}.5^{59}\left(\frac{5^2.41}{2^{10}}\right)^5\left(\frac{2^{20}}{5^2.3.11.31.41}\right)^8\left(\frac{5.2^5.41}{3^8}\right)^3\left(\frac{2^3.3^2.7.31}{5^6}\right)^8\left(\frac{3^4.11^2}{2^3.5^2.7^2}\right)^4,$$

where on the right-hand side the several prime factors have the indices following, viz.

2, index is	(59	+ 160	+ 15	+ 24	− 50	− 12)	= 196,
3	„	(16	+ 16	− 8	− 24	)	= 0,
5	„	(59	+ 10	+ 3	− 16	− 48	− 8)	= 0,
7	„	(8	− 8	)	= 0,
11	„	(8	− 8	)	= 0,
31	„	(8	− 8	)	= 0,
41	„	(5	+ 3	− 8	)	= 0,

or the right-hand side is $=2^{196}$ as it should be. The value of log 2 calculated from $2^{196}=10^{59}$ is $\log 2=\frac{59}{196}=\cdot 301020$, viz. there is an error of a unit in fifth place of decimals. The actual value of 2^{196} has been given by me Mr. Glaisher:

$$2^{196}=10043\ 36277\ 66186\ 89222\ 13726\ 30771$$
$$32266\ 26576\ 37687\ 11142\ 45522\ 06336.*$$

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* The value was deduced from Mr. Shanks's value of 2^{193} in his *Rectification of the Circle*, (1853), p. 90. J. W. L. G.

Supposing $\log 2$ calculated by the form, we then have

$$41 = (\tfrac{10}{1}\tfrac{8}{0}\tfrac{2}{2}\tfrac{5}{4})\, 2^{12} \div 10^5, \text{ giving } \log 41,$$

and

$$3^8 = 10.\tfrac{6}{5}\tfrac{5}{6}\tfrac{6}{5}\tfrac{1}{0}.2^4.41, \text{ giving } \log 3,$$

and formulæ may be obtained proper for the calculation of the logarithms of $\frac{1}{7}$, 11.31, and 7.31.

A. CAYLEY.

Note on a Definite Integral.

The integral $J = \int_0^1 \frac{k^2x^2dx}{\sqrt{(1-x^2.1-k^2x^2)}}$ of Weierstrass is at once seen to be $= K - E$, but the proof that the other integral $J' = \int_1^{\frac{1}{k}} \frac{k^2x^2d\kappa}{\sqrt{(x^2-1.1-k^2x^2)}}$ is $= E'$ is not so immediate.

We have

$$\frac{d}{dy}\frac{y\sqrt{(1-y^2)}}{\sqrt{(1-k^2y^2)}} = \frac{1-2y^2+k^2y^4}{(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{1}{2}}};$$

and thence

$$0 = \int_0^1 \frac{(1-2y^2+k^2y^4)\,dy}{(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{1}{2}}},$$

viz. replacing the numerator by $-\frac{k'^2}{k^2} + \frac{1}{k^2}(1+k^2y^2)^2$ this becomes

$$0 = -\frac{k'^2}{k^2}\int_0^1 \frac{dy}{(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{1}{2}}} + \frac{1}{k^2}\int_0^1 \frac{(1-k^2y^2)^{\frac{1}{2}}dy}{(1-y^2)^{\frac{1}{2}}},$$

that is

$$\int_0^1 \frac{dy}{(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{1}{2}}} = \frac{1}{k'^2}E;$$

or writing k' for k

$$\int_0^1 \frac{dy}{(1-y^2)^{\frac{1}{2}}(1-k'^2y^2)^{\frac{1}{2}}} = \frac{1}{k^2}E'.$$

The integral J' writing therein $x = \frac{1}{\sqrt{(1-k'^2y^2)}}$ becomes

$$J' = k^2\int_0^1 \frac{dy}{(1-y^2)^{\frac{1}{2}}(1-k'^2y^2)^{\frac{1}{2}}},$$

viz. its value is thus $= E'$.

A. CAYLEY.

On a Formula in Elliptic Functions.

Writing $\operatorname{en} u = \dfrac{\operatorname{cn} u}{\operatorname{dn} u}$, then the formulæ p. 63 of my Elliptic Functions give

$$\operatorname{sn}(u+v) = \frac{T - T'}{C - C'}, \quad \operatorname{en}(u+v) = \frac{B + B'}{C - C'};$$

and, substituting for T, T', B, B', and C, C' their values we obtain

$$\operatorname{sn}(u+v) = \frac{\operatorname{sn} u \operatorname{en} v + \operatorname{sn} v \operatorname{en} u}{1 + k^2 \operatorname{sn} u \operatorname{en} u \operatorname{sn} v \operatorname{en} u},$$

$$\operatorname{en}(u+v) = \frac{\operatorname{en} u \operatorname{en} v - \operatorname{sn} u \operatorname{sn} v}{1 - k^2 \operatorname{sn} u \operatorname{en} u \operatorname{sn} v \operatorname{en} v},$$

formulæ which, as regards their numerators, correspond precisely with the formulæ, $\sin(u+v) = \sin u \cos v + \sin v \cos u$ and $\cos(u+v) = \cos u \cos v - \sin u \sin v$, of the circular functions, and which in fact reduce themselves to these on putting $k = 0$.

The foregoing formulæ putting therein $k^2 = -1$, are the formulæ given, Gauss, *Werke*, t. III. p. 404, for the lemniscate functions $\sin \operatorname{lemn}(a \pm b)$ and $\cos \operatorname{lemn}(a \pm b)$; where it is to be observed that these notations do not represent a sine and a cosine, but they are related as the sn and en, viz. that $\cos \operatorname{lemn} a = \sqrt{(1 - \sin \operatorname{lemn}^2 a)} \div \sqrt{(1 + \sin \operatorname{lemn}^2 a)}$.

A. CAYLEY.

Value of a Series.

The series is

$$\frac{1}{1.2.3.4} + \frac{1}{5.6.7.8} + \frac{1}{9.10.11.12} + \frac{1}{13.14.15.16} + \&c. \textit{ ad inf.},$$

the value of which is

$$\tfrac{1}{4} \log 2 - \tfrac{1}{24} \pi.$$

For,

$$\frac{1}{1.2.3.4} = \tfrac{1}{2}\left(\frac{1}{1.4} - \frac{1}{2.3}\right)$$

$$\frac{1}{5.6.7.8} = \tfrac{1}{2}\left(\frac{1}{5.8} - \frac{1}{6.7}\right)$$

$$\frac{1}{9.10.11.12} = \tfrac{1}{2}\left(\frac{1}{9.12} - \frac{1}{10.11}\right),$$

$$\&c. = \&c.,$$

so that the series

$$= \tfrac{1}{2}\left(\frac{1}{1.4} + \frac{1}{5.8} + \frac{1}{9.12} + \&c. \right.$$

$$\left. - \frac{1}{2.3} - \frac{1}{6.7} - \frac{1}{10.11} + \&c. \right)$$

$$= \tfrac{1}{2}(S_1 - S_2),$$

where

$$S_1 = \tfrac{1}{3}(1 - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{8} + \tfrac{1}{9} - \tfrac{1}{12} + \tfrac{1}{13} - \tfrac{1}{16} + \&c.),$$

$$S_2 = \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{6} - \tfrac{1}{7} + \tfrac{1}{10} - \tfrac{1}{11} + \tfrac{1}{14} - \tfrac{1}{15} + \&c.).$$

Now $1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \&c. = \log 2,$

and evidently

$$1 + \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} - \tfrac{1}{7} - \tfrac{1}{8} + \&c. = \tfrac{1}{4}\pi + \tfrac{1}{2}\log 2,$$

whence by addition and subtraction

$$3S_1 = 1 - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{8} + \tfrac{1}{9} - \tfrac{1}{12} + \&c. = \tfrac{1}{8}\pi + \tfrac{3}{4}\log 2,$$

$$S_2 = \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{6} - \tfrac{1}{7} + \tfrac{1}{10} - \tfrac{1}{11} + \&c. = \tfrac{1}{8}\pi - \tfrac{1}{4}\log 2;$$

therefore

$$\tfrac{1}{2}(S_1 - S_2) = \tfrac{1}{2}\{\tfrac{1}{24}\pi + \tfrac{1}{4}\log 2 - \tfrac{1}{8}\pi + \tfrac{1}{4}\log 2\}$$

$$= \tfrac{1}{4}\log 2 - \tfrac{1}{24}\pi.$$

The above series was suggested by the formula

$$\frac{1}{1.3.5.7} + \frac{1}{9.11.13.15} + \frac{1}{17.19.21.23} + \&c. = \frac{\pi}{96(2+\sqrt{2})},$$

which was set in the Senate-House Examination for 1850 (Thursday morning, January 17, Question 6). This formula can be proved by a method very similar to that employed above by means of the equation

$$\frac{\pi}{2\sqrt{2}} = 1 + \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} - \&c.$$

which may be obtained at once by putting $n = 4$ in

$$\pi = n \sin\frac{\pi}{n}\left\{1 + \frac{1}{n-1} - \frac{1}{n+1} - \frac{1}{2n-1} + \frac{1}{2n+1} - \&c.\right\}.$$

A solution is given by Ferrers and Jackson in the *Solutions of the Cambridge Senate-House Problems*, 1848-1851, pp. 226, 227.

J. W. L. GLAISHER.

Note on "Choice and Chance."

The proof of Prop. XLVI. in the 3rd Edition of my *Choice and Chance* is needlessly complicated, and it labours under the disadvantage of depending on Prop. XLV. which itself depends on Prop. XXVII. The following independent proof is much to be preferred.

PROPOSITION XLVI. *If μ be the chance of an operation succeeding in any trial, the chance that in n trials there shall not be k consecutive successes is the coefficient of x^n in*

$$\frac{1-\mu^k x^k}{1-x+(1-\mu)\mu^k x^{k+1}}.$$

Let K_n denote the chance that there shall not be k consecutive successes in n trials. Then, whatever be the result of the first trial, K_{n-1} is the chance that there are not k consecutive successes in the remaining $n-1$ trials; and therefore that there are not k consecutive successes in the n trials *unless* the first k trials be successes, and the $k+1^{\text{th}}$ a failure. But the chance of this case is

$$\mu^k(1-\mu)K_{n-k-1}.$$

Therefore we have

$$K_n = K_{n-1} - (1-\mu)\mu^k K_{n-k-1} \quad \ldots\ldots\ldots\ldots (1).$$

But it is plain that $K_1, K_2 \ldots K_{k-1}$ are each unity, and that $K_k = 1-\mu^k$.

Therefore K_n is the coefficient of x^n in the series

$$S = 1 + x + x^2 + \ldots + x^{k-1} + (1-\mu^k)x^k + \ldots.$$

But from this series we obtain in virtue of (1)

$$\{1-x+(1-\mu)\mu^k x^{k+1}\}S = 1-\mu^k x^k,$$

or

$$S = \frac{1-\mu^k x^k}{1-x+(1-\mu)\mu^k x^{k+1}}.$$

Therefore the required chance is the coefficient of x^n in the expansion of

$$\frac{1-\mu^k x^k}{1-x+(1-\mu)\mu^k x^{k+1}}.$$

Q. E. D.

W. ALLEN WHITWORTH.

Theorem relating to a system of conics.

It is a known theorem, and an old one, that if a system of conics be inscribed on the same quadrilateral, *i.e.* have four common tangents, the pencil of tangents drawn from any point forms a system in involution. The converse of this theorem is also true, *i e.* that if a system of conics be such that the pencil of tangents drawn from any point be in involution, then the system of conics possesses four common tangents.

As in the old theorem it is at once proved that the director circles all pass through two fixed points, and that therefore the centre of the conics all lie on one straight line. Hence it is clear by projection that the locus of the pole of any straight line with respect to the system of conics is a straight line. Hence the coefficients in the tangential equation must contain an arbitrary parameter in the first degree; or the equation to any one of the system must be $U+\lambda V=0$, which proves the theorem.

R. PENDLEBURY.

Note on the Theorem in Kinematics (vol. VII. p. 190).

Prof. Liguine, of Odessa, has pointed out, with reference to the theorem given by me in the *Messenger*, vol. VII. p. 190 and vol. VIII. p. 42,

(1) That the zero-circle may shrink to a point and become imaginary, its radius being $\sqrt{\{(-a^2)\}}$, so that the formula of p. 42 becomes $(P)=N\pi(r^2+a^2)$;

(2) That the centre of the concentric circles is always real and the area described by it a minimum.

A. B. KEMPE.

Note on an Inequality.

If a be greater or less than unity, n being a positive integer, then

$$\frac{a^{2n+2}-1}{a(a^{2n}-1)} > \frac{n+1}{n}.$$

Demonstrations of this proposition have been given in the first series of the *Messenger* by Dr. Ingleby, vol. III, p. 235; by A. Q. G. C., vol. IV. p. 39; by a Student in Professor Kelland's class, vol. IV, p. 109; and by Mr. Sharpe, vol. IV. p. 123. The following proof may be added to the preceding,

Let a be > 1: then

$$\frac{a^{2n+2}-1}{a(a^{2n}-1)} \gtrless \frac{n+1}{n},$$

$$na^{2n+2} - n \gtrless (n+1)a^{2n+1} - (n+1)a,$$

$$na^{2n+1}(a-1) + n(a-1) \gtrless a^{2n+1} - a,$$

$$na^{2n+1} + n \gtrless a^{2n} + a^{2n-1} + \ldots + a,$$

$$a^{2n}(a-1) + a^{2n-1}(a^2-1) + \ldots + a^{n+1}(a^n-1)$$

$$\gtrless \overline{a^n-1} + \overline{a^{n-1}-1} + \overline{a^{n-2}-1} + \ldots + \overline{a-1},$$

$$a^{2n} + a^{2n-1}(a+1) + a^{2n-2}(a^2+a+1) + \ldots + a^{n+1}(a^{n-1}+a^{n-2}+\ldots+1)$$

$$\gtrless \overline{a^{n-1}+a^{n-2}+\ldots+1} + \overline{a^{n-2}+a^{n-3}+\ldots+1} + \ldots + \overline{a+1} + 1,$$

$$a^{2n} - 1 + (a^{2n-1}-1)(a+1) + (a^{2n-2}-1)(a^2+a+1) + \ldots$$

$$+ (a^{n+1}-1)(a^{n-1}+a^{n-2}+\ldots+1) \gtrless 0.$$

Evidently the sign $>$ is the correct one.

Let $a < 1$ and put $a = \frac{1}{\alpha}$, where $\alpha > 1$: then

$$\frac{a^{2n+2}-1}{a(a^{2n}-1)} = \frac{\alpha^{2n+2}-1}{\alpha(\alpha^{2n}-1)},$$

and is therefore, by the preceding proof, $> \frac{n+1}{n}$.

W. WALTON

ON SOME SYMMETRICAL FORMS OF DETERMINANTS.

By *R. F. Scott, M.A.*, St. John's College.

§1. IF

$$D_n = \begin{vmatrix} c, & a, & 0, & 0, & 0, \ldots \\ b, & c, & a, & 0, & 0, \ldots \\ 0, & b, & c, & a, & 0, \ldots \\ 0, & 0, & b, & c, & a, \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{vmatrix},$$

n being the order of the determinant, we get, on expanding

the determinant according to the elements of the first row and column, by Cauchy's theorem

$$D_n = cD_{n-1} - abD_{n-2},$$

an equation of differences for D_n. Solving this in the usual way and determining the constants from the known values of D_1 and D_2, we get

$$D_n = \frac{\{c + \sqrt{(c^2 - 4ab)}\}^{n+1} - \{c - \sqrt{(c^2 - 4ab)}\}^{n+1}}{2^{n+1}\sqrt{(c^2 - 4ab)}}.$$

By giving special values to the quantities a, b, c we get results of more or less interest—

(i) If $c = a + b$

$$D_n = \frac{a^{n+1} - b^{n+1}}{a - b}.$$

(ii) If $c = 2\cos\theta$, $a = b = 1$

$$D_n = \frac{\sin(n+1)\theta}{\sin\theta}.$$

A result given by Prof. Wolstenholme, *Reprints from Ed. Times*, XXI. p. 83.

(iii) If $c = 2\tan\theta$, $a = -1$, $b = 1$,

$$D_n = \frac{(\tan\theta + \sec\theta)^{n+1} - (\tan\theta - \sec\theta)^{n+1}}{2\sec\theta}.$$

(iv) If $a = b = c = 1$

$$D_n = \frac{2}{\sqrt{3}} \sin\frac{\pi(n+1)}{3}.$$

(v) If $c^2 = 4ab$ the solution in the general case becomes

$$D_n = \left(\frac{c}{2}\right)^n (n+1).$$

§ 2. Let

$$D_n = \begin{vmatrix} 0, & k, & k, & k, \ldots \\ 1, & 0, & k, & k, \ldots \\ 1, & 1, & 0, & k, \ldots \\ 1, & 1, & 1, & 0, \ldots \\ \ldots\ldots\ldots\ldots\ldots \end{vmatrix},$$

All the elements in the diagonal being zero, all above it k's, all below it unity.

Subtract the third row from the second, the fourth from the third, and so on, then

$$\frac{D_n}{k} = \begin{vmatrix} 0, & 1, & 1, & 1, \ldots 1 \\ 0, & -1, & k, & 0, \ldots 1 \\ 0, & 0, & -1, & k, \ldots 1 \\ \cdots\cdots & & & \\ 1, & 1, & 1, & 1, \ldots 0 \end{vmatrix}$$

$$= (-1)^{n+1} \begin{vmatrix} 1, & 1, & 1, & 1, \ldots \\ -1, & k, & 0, & 0, \ldots \\ 0, & -1, & k, & 0, \ldots \\ \cdots\cdots & & & \end{vmatrix}_{n-1},$$

or

$$\frac{D_n}{k} = (-1)^{n+1} \left\{ k^{n-2} + \begin{vmatrix} 1, & 1, & 1, \ldots \\ -1, & k, & 0, \ldots \\ 0, & -1, & k, \ldots \\ \cdots\cdots & & \end{vmatrix}_{n-2} \right\}$$

The suffix denoting the order of the determinant. Proceeding in this way

$$D_n = (-1)^{n+1} k \{k^{n-2} + k^{n-3} + \ldots + k + 1\}$$

$$= (-1)^{n-1} k \frac{k^{n-1} - 1}{k - 1}.$$

§ 3. In the determinant of the preceding paragraph multiply every column by b and write $kb = a$. Then we get

$$\begin{vmatrix} 0, & a, & a, & a, \ldots a \\ b, & 0, & a, & a, \ldots a \\ b, & b, & 0, & a, \ldots a \\ \cdots\cdots & & & \\ b, & b, & b, & b, \ldots 0 \end{vmatrix} = (-1)^{n-1} ab \frac{a^{n-1} - b^{n-1}}{a - b}.$$

This expresses a geometrical progression in the form of a determinant.

§ 4. By means of § 3 and Prof. Cayley's theorem for the expansion of a determinant according to products of elements in the principal diagonal (*Crelle*, XXXVIII. p. 93) we can evaluate determinants of this nature when the elements of the principal diagonal do not vanish. We then arrive at the following results:

(i) If all the elements of the principal diagonal are equal to c all above it a, all below it b

$$\begin{vmatrix} c, & a, & a, & a, \ldots \\ b, & c, & a, & a, \ldots \\ b, & b, & c, & a, \ldots \\ b, & b, & b, & c, \ldots \\ \ldots\ldots\ldots\ldots \end{vmatrix} = \frac{a(c-b)^n - b(c-a)^n}{a-b}.$$

If $c = a + b$ this becomes identical with § 1 (1).

(ii) If in the diagonal there are p elements equal to c, q elements equal to d, r elements equal to e, &c. where

$$p+q+r+\ldots = n,$$

then

$$D = \frac{a(c-b)^p(d-b)^q(e-b)^r\ldots - b(c-a)^p(d-a)^q(e-a)^r\ldots}{a-b}.$$

(iii) If the elements of the principal diagonal are $c_1, c_2, \ldots c_n$ and $f(x)$ denotes the continued product

$$(c_1 - x)(c_2 - x)\ldots(c_n - x),$$

then

$$D = \frac{af(b) - bf(a)}{a-b}.$$

By putting $a = b$ and evaluating the vanishing fractions we are led to some known theorems, *e.g.* (iii) gives us the result of Ex. 15, p. 16 of Salmon's *Higher Algebra*. A result due to Mr. Michael Roberts, *Nouv. Ann.*, 1864.

§ 5. In the same way in which the result of § 2 was obtained we may prove that

$$\begin{vmatrix} 0, & a_2, & a_3, & a_4, & a_5, \ldots \\ b_1, & 0, & a_3, & a_4, & a_5, \ldots \\ b_1, & b_2, & 0, & a_4, & a_5, \ldots \\ b_1, & b_2, & b_3, & 0, & a_5, \ldots \\ b_1, & b_2, & b_3, & b_4, & 0, \ldots \\ \ldots\ldots\ldots\ldots\ldots \end{vmatrix} = (-1)^{n-1}[b_1a_2a_3\ldots a_n + b_1b_2a_3\ldots a_n + b_1b_2b_3a_4\ldots a_n + \ldots + b_1b_2\ldots\ldots a_n],$$

where the law of formation is obvious.

By giving to the a's and b's different values we get numerous results.

By filling in the diagonal we get results analogous to § 4, but of greater complexity.

§ 6. From § 4 (ii), if we put all the elements of the diagonal equal to c, except the last, which is put equal to zero, we get, after dividing both sides by ab,

$$\begin{vmatrix} c, & a, & \ldots a, & 1 \\ b, & c, & \ldots a, & 1 \\ \ldots\ldots & & & \\ b, & b, & \ldots c, & 1 \\ 1, & 1, & \ldots 1, & 0 \end{vmatrix} = \frac{(c-a)^{n-1} - (c-b)^{n-1}}{a-b},$$

a result we shall find useful in what follows.

There is a corresponding result for § 4 (iii).

§ 7. It is, of course, clear that the reason why the determinants in the preceding paragraphs reduce to simple forms is the symmetry with respect to the principal diagonal. It is a natural step then to inquire if there are any similar results when the symmetry which exists is with respect to both diagonals.

As a first step in this direction, consider the determinant

$$\begin{vmatrix} a, & c, & c, & c, & c, & b \\ c, & a, & c, & c, & b, & c \\ c, & c, & a, & b, & c, & c \\ c, & c, & b, & a, & c, & c \\ c, & b, & c, & c, & a, & c \\ b, & c, & c, & c, & c, & a \end{vmatrix}$$

of order $2n$.

If $a = b$, the first column is the same as the last, the second the same as the last but one, and so one; hence the determinant divides by $(a-b)^n$.

Adding all the columns to the first, each element becomes $[a+b+2c(n-1)]$, which is therefore a factor of the determinant.

Add the last column to the first, the last but one to the second, and so on, then the first n columns would become identical, viz. all their elements would be $2c$, if $a+b=2c$, whence $(a+b-2c)^{n-1}$ is a factor of the determinant. Thus the determinant divides by the product

$$(a-b)^n (a+b-2c)^{n-1} \{a+b+2c(n-1)\}.$$

This being of the same order as the determinant, and the coefficients of a^{2n} being the same in both, the determinant must be equal to the product.

§8. On looking at the determinant in §7, we see that what we have now got is symmetry with regard to the centre of the square, in which the elements of the determinant are written. Also that the form of the determinant is different according as its order is $2n$ or $2n+1$.

For the present let us confine ourselves to the determinants formed as follows: Draw the two diagonals of a square and two lines through its centre parallel to the sides, thus dividing it into eight triangles. All the elements in each triangle are to be the same, and pairs of triangles symmetrically situated with regard to the centre are to have the same elements; all elements in the principal diagonal are to be x in the other y. The other dividing lines will or will not contain elements according as the order of the determinant is $2n+1$ or $2n$.

§9. Evaluation of the determinants

$$\begin{vmatrix} x, & a, & a, & b, & b, & y \\ d, & x, & a, & b, & y, & c \\ d, & d, & x, & y, & c, & c \\ c, & c, & y, & x, & d, & d \\ c, & y, & b, & a, & x, & d \\ y, & b, & b, & a, & a, & x \end{vmatrix} \quad \ldots(1),$$

$$\begin{vmatrix} x, & a, & a, & p, & b, & b, & y \\ d, & x, & a, & p, & b, & y, & c \\ d, & d, & x, & p, & y, & c, & c \\ q, & q, & q, & r, & q, & q, & q \\ c, & c, & y, & p, & x, & d, & d \\ c, & y, & b, & p, & a, & x, & d \\ y, & b, & b, & p, & a, & a, & x \end{vmatrix} \quad \ldots(2),$$

of order $2n$ and $2n+1$.

In (1) subtract the last column from the first, the last but one from the second, and so on. Then in the determinant so transformed add the first row to the last, the second to the last but one, and so on. Then all the elements in the square block, which in (1) contains the elements

$$\begin{vmatrix} c, & c, & y \\ c, & y, & b \\ y, & b, & b \end{vmatrix}$$

vanish, and the determinant reduces to the product of the two determinants

$$\begin{vmatrix} x-y, & a-b, & a-b \\ d-c, & x-y, & a-b \\ d-c, & d-c, & x-y \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} x+y, & c+d, & c+d \\ a+b, & x+y, & c+d \\ a+b, & a+b, & x+y \end{vmatrix},$$

both of which are of the form (i) in §4.

Thus the value of the determinant (1) is the product of the two expressions

$$\left.\begin{array}{l} \dfrac{(a-b)(x-y+c-d)^n-(d-c)(x-y-a+b)^n}{a-b+c-d} \\ \dfrac{(a+b)(x+y-c-d)^n-(c+d)(x+y-a-b)^n}{a+b-c-d} \end{array}\right\} \ldots\ldots (3).$$

Pursuing the same process of reduction with the determinant (2), we reduce it to the product

$$\begin{vmatrix} x-y, & a-b, & a-b \\ d-c, & x-y, & a-b \\ d-c, & d-c, & x-y \end{vmatrix} \text{ and } \begin{vmatrix} r, & q, & q, & q \\ 2p, & x+y, & c+d, & c+d \\ 2p, & a+b, & x+y, & c+d \\ 2p, & a+b, & a+b, & x+y \end{vmatrix},$$

the first of which is of the form § 4 (i); the second is the sum of two determinants, one of the form § 4 (i), the other of the form § 6.

Thus (2) reduces to the product of the expressions

$$\frac{(a-b)(x-y+c-d)^n-(d-c)(x-y-a+b)^n}{a-b+c-d},$$

$$\left[r\frac{(a+b)(x+y-c-d)^n-(c+d)(x+y-a-b)^n}{a+b-c-d} + 2pq\frac{(x+y-a-b)^n-(x+y-c-d)^n}{a+b-c-d}\right].$$

§ 10. If the elements in the diagonals are not all equal but only equal in pairs equidistant from the centre we can evaluate the resulting determinants of § 9 by § 4 (iii).

The property possessed by the above of being resolvable into the product of two determinants is common to all determinants symmetrical about the centre.

By introducing special relations between the quantities involved a great number of special cases may be deduced, some of which are interesting, but it seems unnecessary to enumerate them here.

§ 11. The determinants of § 9 may now be extended.

These determinants are formed from that in § 4 (1) by taking the two determinants

$$\begin{vmatrix} x, & a, & a \\ d, & x, & a \\ d, & d, & x \end{vmatrix}, \quad \begin{vmatrix} b, & b, & y \\ b, & y, & c \\ y, & c, & c \end{vmatrix},$$

and forming a new square array, by repeating each twice so as to get § 9 (1).

But we can take two determinants of the form § 9 (1) and repeating each twice obtain a new determinant of order $4n$. This again, by the same transformation, as above can be resolved into the product of two determinants of the form § 9 (1), and hence its value is known, viz. if the elements of the two determinants of the form § 9 (1) are denoted by suffixes, the value of the new determinant is the product of four factors, the first and second of which are got from § 9 (3) by writing for x, y, a, b, c, d; $x_1 - x_2$, $y_1 - y_2$, $a_1 - a_2$, &c., and the third and fourth by writing $x_1 + x_2$, $y_1 + y_2$, $a_1 + a_2$, &c.

We get determinants of order $4n+1$, $4n+2$, $4n+3$ by taking a determinant of the form (1) with another of the form (2), or two of the form (2), or two of the form (2) with a dividing line, all of which can be evaluated in like manner.

We may then extend these new determinants, and so on interminably.

§ 12. By interchanging the order of the rows and columns we get from the determinants written down other forms of symmetry. For example, from that in § 7 by removing the first row to the last place continually n times we change the determinant into one where all the elements are the same except those lying on the sides of a square inscribed in the square of the original elements, the angular points of the second square being at the middle points of the sides of the first. Similar remarks apply to the others.

NOTE ON CERTAIN DETERMINANTS CONNECTED WITH ALGEBRAICAL EXPRESSIONS HAVING THE SAME FORM AS THEIR COMPONENT FACTORS.

By *Samuel Roberts, M.A., F.R.S.*

If we assume Lagrange's equivalence

$$\begin{aligned}(x^2 + ax_1^2 + bx_2^2 + abx_3^2)\,(X^2 + aX_1^2 + bX_2^2 + abX_3^2) \\ = \quad \{\, xX + ax_1X_1 + bx_2X_2 + abx_3X_3\}^2 \\ + a\,\{-xX_1 + x_1X - bx_2X_3 + bx_3X_2\}^2 \\ + b\,\{-xX_2 + ax_1X_3 + x_2X - ax_3X_1\}^2 \\ + ab\,\{-xX_3 - x_1X_2 + x_2X_1 + x_3X\}^2,\end{aligned}$$

it is plain that the determinant

$$\begin{vmatrix} +X, & +aX_1, & +bX_2, & +abX_3 \\ -X_1, & +X, & -bX_3, & +bX_2 \\ -X_2, & +aX_3, & +X, & -aX_1 \\ -X_3, & -X_2, & +X_1, & +X \end{vmatrix},$$

must be a power of $(X^2+aX_1^2+bX_2^2+abX_3^2)$ to a constant factor *près*. The power is the second and the constant is unity, as appears by inspection.

In like manner, if we consider the corresponding equivalence for forms of 8 terms whose coefficients are the terms of $(1+a)(1+b)(1+c)$, an analogous determinant of 64 elements is arrived at. It is not necessary to write down the extended equivalence in full. We have

$$\begin{aligned}&(x^2+ax_1^2+abx_2^2+bx_3^2+acx_4^2+cx_5^2+bcx_6^2+abcx_7^2)\\ &\times(X^2+aX_1^2+abX_2^2+bX_3^2+acX_4^2+cX_5^2+bcX_6^2+abcX_7^2)\\ =\;&P^2+aP_1^2+abP_2^2+bP_3^2+acP_4^2+cP_5^2+bcP_6^2+abcP_7^2,\end{aligned}$$

where $P, P_1, \ldots P_7$ are linear and homogeneous in $x, x_1, \ldots x_7$ and $X, X_1, \ldots X_7$.

If, then, we form the condition in $X, X_1, \ldots X_7$ that $P, P_1, \ldots P_4$ may vanish, the function must be a power of the second factor of the left-hand member of the above equation to a constant factor *près*. The power is seen to be the fourth and the constant is unity for the form of determinant here given, namely:

$$\begin{vmatrix} +X, & +aX_1, & +abX_2, & +bX_3, & +acX_4, & +cX_5, & +bcX_6, & +abcX_7 \\ -X_1, & +X, & +bX_3, & -bX_2, & +cX_5, & -cX_4, & -bcX_7, & +bcX_6 \\ -X_2, & -aX_3, & +X, & +X_1, & +cX_6, & +cX_7, & -cX_4, & -cX_5 \\ -X_3, & +aX_2, & -aX_1, & +X, & +acX_7, & -cX_6, & +cX_5, & -acX_4 \\ -X_4, & -X_5, & -bX_6, & -bX_7, & +X, & +X_1, & +bX_2, & +bX_3 \\ -X_5, & +aX_4, & -abX_7, & +bX_6, & -aX_1, & +X, & -bX_3, & +abX_2 \\ -X_6, & +aX_7, & +aX_4, & -X_5, & -aX_2, & +X_3, & +X, & -aX_1 \\ -X_7, & -X_6, & +X_5, & +X_4, & -X_3, & -X_2, & +X_1, & +X \end{vmatrix}.$$

The forms $P, P_1, \ldots P_7$ are found by multiplying each element of the first column by x, of the second by x_1, and so forth.*

* See another expression for the case of $a=b=c=1$ in Mr. J. J. Thomson's paper, "On the resolution of a product of two sums of eight squares into the sum of eight squares," vol. VII. pp. 73, 74.

The determinant may, of course, be varied (1) by writing negatively one or more of $X, X_1, \ldots X_7$; (2) by interchanging columns with or without change of sign in all the terms; (3) by similarly interchanging rows, operations which may be effected simultaneously.

If in the determinant we change the signs of $X_1, X_2, \ldots X_7$, leaving the diagonal unchanged, and then convert the first column into the top row, by turning it about X till X_7 takes the place of $abcX_7$, the second column becoming the second row in a similar way, and so on, the determinant thus obtained (having the same value) will, when multiplied into the original determinant by the usual rule, give

$$\begin{array}{c} \Phi, 0, 0, 0, 0, 0, 0, 0, \\ 0, \Phi, 0, 0, 0, 0, 0, 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0, 0, 0, 0, 0, 0, 0, \Phi, \end{array}$$

where Φ is second component of the left-hand member of the equivalence.

In fact, each of the principal minor determinants contains Φ as a factor, and the reciprocal determinant is of similar form.

There are then analogous determinants of 4 elements, 16 elements, and 64 elements.

The continuance of the series depends on the extension of Lagrange's equivalence to similar forms of 2^n terms. But the literature on this subject is singularly conflicting, and seems to require critical examination.

THEOREMS IN ALGEBRA.

By *J. W. L. Glaisher.*

I.

§ 1. If $a^2 + c^2 - 2bd = 1$ and $b^2 + d^2 - 2ac = 0$, then

$$\begin{vmatrix} a, & b, & c, & d \\ d, & a, & b, & c \\ c, & d, & a, & b \\ b, & c, & d, & a \end{vmatrix} = 1,$$

and in this determinant

$$a=\begin{vmatrix} a, & b, & c \\ d, & a, & b \\ c, & d, & a \end{vmatrix}, \quad b=\begin{vmatrix} b, & c, & d \\ d, & a, & b \\ c, & d, & a \end{vmatrix}, \quad c=\begin{vmatrix} b, & c, & d \\ a, & b, & c \\ c, & d, & a \end{vmatrix}, \quad d=\begin{vmatrix} b, & c, & d \\ a, & b, & c \\ d, & a, & b \end{vmatrix},$$

i.e. the r^{th} element in the first row $=(-)^{r+1}$ the minor of the r^{th} element in the first column.

This is easily proved for the determinant

$$(a^2+c^2-2bd)^2-(b^2+d^2-2ac)^2=1,$$

and also

$$aa-db+cc-bd=1,$$

$$ba-ab+dc-cd=0,$$

$$ca-bb+ac-dd=0,$$

$$da-cb+bc-ad=0,$$

the first and third of these being the given equations, and the second and fourth identities. Writing down from these equations the values of a, b, c, d as the quotients of two determinants, and observing that the determinant which forms the denominator is equal to unity we obtain the second set of relations.

§2. If

$$a^2-d^2+2ec-2bf=1,$$

$$e^2-b^2+2ca-2fd=0,$$

$$c^2-f^2+2ae-2db=0,$$

then

$$\begin{vmatrix} a, & b, & c, & d, & e, & f \\ f, & a, & b, & c, & d, & e \\ e, & f, & a, & b, & c, & d \\ d, & e, & f, & a, & b, & c \\ c, & d, & e, & f, & a, & b \\ b, & c, & d, & e, & f, & a \end{vmatrix}=1,$$

and if A, F, E, D, C, B denote the minors of a, f, e, d, c, b in the first column, then

$$a=A,\ b=-F,\ c=E,\ d=-D,\ e=C,\ f=-B.$$

§ 3. In general, if the $2n$ quantities $a_1, a_2, \ldots a_{2n}$ be connected by the n relations

$$\begin{aligned}
&a_1a_1 \quad - a_2a_{2n} \quad + a_3a_{2n-1} \ldots - a_{2n}a_2 = 1,\\
&a_1a_3 \quad - a_2a_2 \quad + a_3a_1 \quad \ldots - a_{2n}a_4 = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&a_1a_{2n-1} - a_2a_{2n-2} + a_3a_{2n-3} \ldots - a_{2n}a_{2n} = 0,
\end{aligned}$$

then

$$\left|\begin{array}{l}
a_1 \ , a_2 \ , a_3, \ldots a_{2n}\\
a_{2n} \ , a_1 \ , a_2, \ldots a_{2n-1}\\
a_{2n-1}, a_{2n}, a_1, \ldots a_{2n-2}\\
\ldots\ldots\ldots\ldots\ldots\ldots\\
a_2 \ , a_3 \ , a_4, \ldots a_1
\end{array}\right| = 1 \ldots\ldots\ldots\ldots \text{(i)},$$

and if $A_1, A_{2n}, A_{2n-1}, \ldots A_2$ denote the minors of $a_1, a_{2n}, \ldots a_2$ in the first column, then

$$a_r = (-)^{r+1} A_{2n-r+2}.$$

The law of formation of the given relations is, that the suffixes of the first factors in the terms run forwards from 1 to $2n$, and those of the second factors run backwards round the cycle, beginning with $2r-1$ in the r^{th} relation.

It can be shown that if we denote the n expressions which form the left-hand members of these equations by $\alpha_1, \alpha_2, \ldots \alpha_n$, then the determinant in (i) is identically equal to the determinant of n rows,

$$\left|\begin{array}{l}
\alpha_1 \ , \alpha_2, \alpha_3, \ldots \alpha_n\\
\alpha_n \ , \alpha_1, \alpha_2, \ldots \alpha_{n-1}\\
\alpha_{n-1}, \alpha_n, \alpha_1, \ldots \alpha_{n-2}\\
\alpha_2 \ , \alpha_3, \alpha_4, \ldots \alpha_1
\end{array}\right|,$$

and when $\alpha_1 = 1, \ \alpha_2 = 0, \ \alpha_3 = 0, \ \ldots \alpha_n = 0$, this determinant obviously $= 1$.

For example, if $n = 3$,

$$\left|\begin{array}{llllll}
a, & b, & c, & d, & e, & f\\
f, & a, & b, & c, & d, & e\\
e, & f, & a, & b, & c, & d\\
d, & e, & f, & a, & b, & c\\
c, & d, & e, & f, & a, & b\\
b, & c, & d, & e, & f, & a
\end{array}\right| = \left|\begin{array}{lll}
\alpha, & \beta, & \gamma\\
\gamma, & \alpha, & \beta\\
\beta, & \gamma, & \alpha
\end{array}\right|,$$

where
$$\alpha = a^2 - d^2 + 2ec - 2bf,$$
$$\beta = e^2 - b^2 + 2ca - 2fd,$$
$$\gamma = c^2 - f^2 + 2ae - 2db,*$$

The relations $a_r = (-)^{r+1} A_{2n-r-2}$ are obtained, as in § 1, by introducing between the n given relations the n identical relations

$$a_1 a_2 - a_2 a_1 + a_3 a_{2n} \ldots - a_{2n} a_3 = 0,$$
$$a_1 a_4 - a_2 a_3 + a_3 a_2 \ldots - a_{2n} a_5 = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The above properties occur in a paper "On a special form of determinant, and on functions of n variables analogous to the sine and cosine" (*Quarterly Journal*, t. XVI. pp. 15–33), but they seem worth stating as mere algebraical results, apart from the notations with which they are connected in that paper.

II.

If a, b, c, a', b', c' be any six quantities, then

$$\begin{aligned}
&(\ 2a+\ b+\ c+\ .\ -\ b'+\ c')^2 \\
&+(\ \ a+2b+\ c+\ a'-\ .\ -\ c')^2 \\
&+(\ \ a+\ b+2c-\ a'+\ b'\ \ .\ )^2 \\
&+(\ .\ +\ b-\ c+2a'-\ b'-\ c')^2 \\
&+(-\ a\ \ .\ +\ c-\ a'+2b'-\ c')^2 \\
&+(\ \ a-\ b\ \ .\ -\ a'-\ b'+2c')^2 \ \ldots\ldots\ldots (1), \\
&= 8\,(a^2\ +b^2\ +c^2\ +a'^2+b'^2+c'^2 \\
&+\ \ bc\ +ca\ +ab\ +b'c+c'a+a'b \\
&-\ \ b'c'-c'a'-a'b'-bc'-ca'-ab') \ \ldots\ldots\ldots (2), \\
&= 4\{(b+c+b'-c')^2+(c+a+c'-a')^2+(a+b+a'-b')^2\} \ldots (3).
\end{aligned}$$

In the expression (2), all the product terms appear except aa', bb', cc'.

It follows from (2), that

$$a^2+b^2+c^2+a'^2+b'^2+c'^2+bc+ca+ab$$
$$+b'c+c'a+a'b > b'c'+c'a'+a'b'+bc'+ca'+ab'.$$

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* By means of this theorem, I have calculated the value of the determinant on the left-hand side, *Quarterly Journal*, t. XVI. p. 33.

By changing the signs of letters in this inequality, we may reduce the number of product terms on the left-hand side to four (but not to *any* four), *e.g.* changing the sign of a,

$$a^2+b^2+c^2+a'^2+b'^2+c'^2+bc+b'c$$
$$+a'b+ab'>ca+ab+b'c'+c'a'+a'b'+bc'+ca'+c'a.$$

The equality of (1) and (3) is evident in virtue of the equation

$$(A+B)^2+(A-B)^2=2(A^2+B^2).$$

I met with the system of squares in (1), in the calculation of the sum of the squares of the residuals in the solution of the equations

$$\left.\begin{array}{llll} x=a, & y=b, & z=c & \\ x+y=d, & y+z=e, & x+y+z=f & \end{array}\right\}\ldots\ldots\ldots(4),$$

by the method of least squares.*

The most probable values of x, y, z are

$$x=\tfrac{1}{4}(\ 2a-\ b+d-\ e+f),$$
$$y=\tfrac{1}{4}(-\ a+2b-\ c+d+e),$$
$$z=\tfrac{1}{4}(-\ b+2c-\ d+\ e+f),$$

and the residuals $x-a$, $y-b$, $\ldots x+y+z-f$ are

$$\tfrac{1}{4}(-2a-\ b\ .\ +\ d-\ e+\ f),$$
$$\tfrac{1}{4}(-\ a-2b-\ c+\ d+\ e\ .\ ),$$
$$\tfrac{1}{4}(\ .\ -\ b-2c-\ d+\ e+f),$$
$$\tfrac{1}{4}(\ \ a+\ b-\ c-2d\ .\ +f),$$
$$\tfrac{1}{4}(-\ a+\ b+\ c\ .\ -2e+f),$$
$$\tfrac{1}{4}(\ \ a\ .\ +\ c+\ d+\ e-2f),$$

viz. the squares of these are the same as the squares in (1) if for a, b, c, a', b', c' in (1) be substituted c, b, $-e$, a, f, d. It was remarked that in the expression for the sum of the squares of the residuals all the coefficients were $\frac{1}{2}$ or $-\frac{1}{2}$, thus giving in effect the equation (1) = (2). It is curious that the unsymmetrical equations (4) should lead to the symmetrical system of squares (1).

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* The complete solution is given in the *Solutions of the Senate-House Problems and Riders for* 1878, pp. 165, 166.

ON SOME SYMMETRICAL FORMS OF DETERMINANTS.

By *R. F. Scott, M.A.*

(Continued from p. 138).

§ 13. Evaluation of the determinant

$$D = \begin{vmatrix} c, & a, & c, & c, & c, & c, \dots \\ b, & c, & a, & c, & c, & c, \dots \\ c, & b, & c, & a, & c, & c, \dots \\ c, & c, & b, & c, & a, & c, \dots \\ c, & c, & c, & b, & c, & a, \dots \\ c, & c, & c, & c, & b, & c, \dots \\ \dots\dots\dots\dots\dots\dots\dots \end{vmatrix},$$

where all the elements are c with the exception of those in the two lines one on either side of the principal diagonal.

This determinant

$$D = \begin{vmatrix} 1, & 0, & 0, & 0, & 0, \dots \\ 1, & c, & a, & c, & c, \dots \\ 1, & b, & c, & a, & c, \dots \\ 1, & c, & b, & c, & a, \dots \\ 1, & c, & c, & b, & c, \dots \\ \dots\dots\dots\dots\dots\dots\dots \end{vmatrix}.$$

Multiply the first column by c and subtract it from each of the others,

$$D = \begin{vmatrix} 1, & -c, & -c, & -c, \dots \\ 1, & 0, & a-c, & 0, \dots \\ 1, & b-c, & 0, & a-c, \dots \\ 1, & 0, & b-c, & 0, \dots \\ 1, & 0, & 0, & b-c, \dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \end{vmatrix}$$

$$= \begin{vmatrix} 0, & x, & 0, & 0, \dots \\ y, & 0, & x, & 0, \dots \\ 0, & y, & 0, & x, \dots \\ 0, & 0, & y, & 0, \dots \\ \dots\dots\dots\dots\dots \end{vmatrix} - c \begin{vmatrix} 0, & 1, & 1, & 1, & 1, \dots \\ 1, & 0, & x, & 0, & 0, \dots \\ 1, & y, & 0, & x, & 0, \dots \\ 1, & 0, & y, & 0, & x, \dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \end{vmatrix};$$

where, for brevity, $x = a - c$, $y = b - c$.

We have then to evaluate these two determinants.

If
$$D_n = \begin{vmatrix} 0, & x, & 0, & 0 \\ y, & 0, & x, & 0 \\ 0, & y, & 0, & x \\ 0, & 0, & y, & 0 \end{vmatrix} = -xyD_{n-2},$$

since
$$D_2 = -xy, \quad D_3 = 0,$$
$$D_{2n} = (-xy)^n, \quad D_{2n+1} = 0.$$

In the evaluation of the second determinant we shall also find it convenient to discriminate between the two cases of even or odd degree.

First let the degree be even and let

$$D_{2n} = \begin{vmatrix} 0, & 1, & 1, & 1, & 1, & 1, & 1, & 1 \\ 1, & 0, & x, & 0, & 0, & 0, & 0, & 0 \\ 1, & y, & 0, & x, & 0, & 0, & 0, & 0 \\ 1, & 0, & y, & 0, & x, & 0, & 0, & 0 \\ 1, & 0, & 0, & y, & 0, & x, & 0, & 0 \\ 1, & 0, & 0, & 0, & y, & 0, & x, & 0 \\ 1, & 0, & 0, & 0, & 0, & y, & 0, & x \\ 1, & 0, & 0, & 0, & 0, & 0, & y, & 0 \end{vmatrix}.$$

Expand by Cauchy's theorem (Salmon, *Higher Algebra*, Art. 36) with regard to the elements of the last row and column,

$$D_{2n} = -xyD_{n-2} - (xy)^{n-1} - x \begin{vmatrix} 1, & 1, & 1, & 1, & 1, & 1 \\ 0, & x, & 0, & 0, & 0, & 0 \\ y, & 0, & x, & 0, & 0, & 0 \\ 0, & y, & 0, & x, & 0, & 0 \\ 0, & 0, & y, & 0, & x, & 0 \\ 0, & 0, & 0, & y, & 0, & x \end{vmatrix}$$

$$- y \begin{vmatrix} 1, & 0, & x, & 0, & 0, & 0 \\ 1, & y, & 0, & x, & 0, & 0 \\ 1, & 0, & y, & 0, & x, & 0 \\ 1, & 0, & 0, & y, & 0, & x \\ 1, & 0, & 0, & 0, & y, & 0 \\ 1, & 0, & 0, & 0, & 0, & y \end{vmatrix},$$

To evaluate the determinant multiplying x observe that it is of order $2n-2$. As it stands there is only one element in the second row, erase the row and column in which it stands, in the resulting determinant there will be only one element in the third row, erase the row and column in which it stands. Proceed in this way and the determinant

$$= x^{n-1} \begin{vmatrix} 1, & 1, & 1, \dots \\ y, & x, & 0, \dots \\ 0, & y, & x, \dots \\ \dots\dots\dots\dots & & \end{vmatrix}_{n-1}.$$

Let $$\delta_n = \begin{vmatrix} 1, & 1, & 1, & 1, \dots \\ y, & x, & 0, & 0, \dots \\ 0, & y, & x, & 0, \dots \\ 0, & 0, & y, & x, \dots \\ \dots\dots\dots\dots & & & \end{vmatrix}_n = x^{n-1} - y\delta_{n-1};$$

therefore $$\delta_n + y\delta_{n-1} = x^{n-1},$$

or $$\delta_n = \frac{x^n - (-y)^n}{x+y}.$$

Thus the determinant multiplying x is

$$x^{n-1}\frac{x^n - (-y)^n}{x+y}.$$

Similarly, that multiplying y is

$$y^{n-1}\frac{y^{n-1} - (-x)^{n-1}}{x+y}.$$

Hence $$D_{2n} = -xyD_{2n-2} - (xy)^n - \frac{x^n}{x+y}\{x^{n-1} - (-y)^{n-1}\} - \frac{y^n}{x+y}\{y^{n-1} - (-x)^{n-1}\},$$

$$D_{2n} = -xyD_{2n-2} - \frac{x^{2n-1} + y^{2n-1}}{x+y}.$$

An equation of differences which gives

$$D_{2n} = A(-xy)^n - \frac{x^{2n} + y^{2n}}{(x+y)^2}.$$

But $$D_4 = -(x-y)^2;$$

therefore $$D_{2n} = 2\frac{(-xy)^n - x^{2n} - y^{2n}}{(x+y)^2} = -\left\{\frac{x^n - (-y)^n}{x+y}\right\}^2.$$

Next, for determinants of odd order we can establish in the same way the equation

$$D_{2n+1} = -xyD_{2n-1} + \frac{x^{2n}+y^{2n}}{x+y} - \frac{2(-xy)^n}{x+y}.$$

Solving this, it appears that

$$D_{2n+1} = A(-xy)^n + \frac{x^{2n+1}+y^{2n+1}}{(x+y)^2} - \frac{2(n+1)(-xy)^n}{x+y}.$$

But $D_3 = x+y$, thus

$$D_{2n+1} = \frac{x^{2n+1}+y^{2n+1}-(2n+1)(x+y)(-xy)^n}{(x+y)^2}.$$

Then, returning to the original determinant, its value is

(i) if of order $2n$,

$$\{(a-c)(c-b)\}^n - c\left[\frac{\{(a-c)^{2n+1}+(b-c)^{2n+1}-(2n+1)\{(a-c)(c-b)\}^n(a+b-2c)\}}{(a+b-2c)^2}\right];$$

(ii) if of order $2n-1$,

$$c\left\{\frac{(a-c)^n-(c-b)^n}{a+b-2c}\right\}^2.$$

§14. Evaluation of the determinant

$$D = \begin{vmatrix} a, & b, & b, & b, \ldots \\ c, & b, & c, & c, \ldots \\ a, & a, & c, & a, \ldots \\ \ldots\ldots\ldots\ldots \end{vmatrix}$$

of order $3n$. The elements in the diagonal consist of a, b, c repeated n times in this order, and the rest of the row is filled up with the next letter in cyclical order,

$$D = \begin{vmatrix} 1, & 1, & 1, & 1, & 1, \ldots \\ 0, & a, & b, & b, & b, \ldots \\ 0, & c, & b, & c, & c, \ldots \\ 0, & a, & a, & c, & a, \ldots \\ \ldots\ldots\ldots\ldots\ldots \end{vmatrix}$$

$$= \begin{vmatrix} 1, & 1, & 1, & 1, & \ldots \\ -b, & a-b, & 0, & 0, & \ldots \\ -c, & 0, & b-c, & 0, & \ldots \\ -a, & 0, & 0, & c-a, & \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{vmatrix}$$

by subtracting the first row multiplied by b, c, or a from the others.

Thus, by Cauchy's theorem,

$$D=[a-b\,.\,b-c\,.\,c-a]^n\left\{1+n\left(\frac{b}{a-b}+\frac{c}{b-c}+\frac{a}{c-a}\right)\right\}.$$

§ 15. In like manner

$$\begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & a, & b, & b, & \dots \\ 1, & c, & b, & c, & \dots \\ 1, & a, & a, & c, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = n[a-b.b-c.c-a]^{n-1}(a^2+b^2+c^2-bc-ca-ab).$$

The determinant is of order $3n+1$, and, with the exception of the border, its formation is the same as of that in § 14.

The determinants of the present and preceding paragraphs can plainly be generalized.

§ 16. In like manner

$$\begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & x, & a_2, & a_3, & \dots \\ 1, & a_1, & x, & a_3, & \dots \\ 1, & a_1, & a_2, & x, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = -\frac{du}{dx},$$

when $u=(x-a_1)(x-a_2)\dots(x-a_n)$.

$$\begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & a_1, & x, & x, & \dots \\ 1, & x, & a_2, & x, & \dots \\ 1, & x, & x, & a_3, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = (-1)^n\frac{du}{dx},$$

$$\begin{vmatrix} x, & a_2, & a_3, & \dots \\ a_1, & x, & a_3, & \dots \\ a_1, & a_2, & x, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = u-\Sigma a_i\frac{du}{da_i}.$$

§ 17. This artifice of adding on a border evaluates all determinants when the elements in a row or column are all the same with the exception of those in the diagonal.

Thus

$$\begin{vmatrix} (x-a_1)^2, & a_2^2, & a_3^2, \ldots \\ a_1^2, & (x-a_2)^2, & a_3^2, \ldots \\ a_1^2, & a_2^2, & (x-a_3)^2, \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{vmatrix} = x^{n-1}u\left\{x+\Sigma\frac{a_i^2}{x-2a_i}\right\},$$

$$\begin{vmatrix} 0, & 1, & 1, & 1, \ldots \\ 1, & (x-a_1)^2, & a_2^2, & a_3^2, \ldots \\ 1, & a_1^2, & (x-a_2)^2, & a_3^2, \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{vmatrix} = -x^{n-1}\frac{du}{dx},$$

where $u=(x-2a_1)(x-2a_2)\ldots(x-2a_n)$.

For example, by the latter theorem,

$$\begin{vmatrix} 0, & 1, & 1, & 1 \\ 1, & (y+z)^2, & y^2, & z^2 \\ 1, & x^2, & (x+z)^2, & z^2 \\ 1, & x^2, & y^2, & (x+z)^2 \end{vmatrix} = (x+y+z)^2(x^2+y^2+z^2 - 2xy-2yz-2zx).$$

If $s=x+y+z+u$,

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & (s-x)^2, & y^2, & z^2, & u^2 \\ 1, & x^2, & (s-y)^2, & z^2, & u^2 \\ 1, & x^2, & y^2, & (s-z)^2, & u^2 \\ 1, & x^2, & y^2, & z^2, & (s-u)^2 \end{vmatrix} = s^3\{x^2(y+z+u)+y^2(x+z+u) +z^2(x+y+u)+u^2(x+y+z) +2xyz+2xzu+2yzu+2xyz -x^3-y^3-z^3-u^3\}.$$

§18. If the element at the intersection of the i^{th} row and j^{th} column be a_ia_j while the diagonal element is as before,

$$\begin{vmatrix} (x-a_1)^2, & a_1a_2, & a_2a_3, \ldots \\ a_1a_2, & (x-a_2)^2, & a_2a_3, \ldots \\ a_1a_3, & a_2a_3, & (x-a_3)^2, \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{vmatrix} = x^{n-1}u\left\{x+\Sigma\frac{a_i^2}{x-2a_i}\right\}$$

$$\begin{vmatrix} 0, & a_1, & a_2, & a_3, \ldots \\ a_1, & (x-a_1)^2, & a_1a_2, & a_1a_3, \ldots \\ a_2, & a_1a_2, & (x-a_2)^2, & a_2a_3, \ldots \\ a_3, & a_1a_3, & a_2a_3, & (x-a_3)^2, \ldots \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{vmatrix} = -x^{n-1}u\Sigma\frac{a_i^2}{x-2a_i},$$

SOLUTION OF A MECHANICAL PROBLEM

By Professor *A. G. Greenhill, M.A.*

THE following rider, due to Mr. Hopkinson, was set in the Mathematical Tripos of 1879, Thursday morning, January 2, Number viii.

"A smooth wire is bent into the form of a circle radius a, and rotates with uniform angular velocity ω about a vertical axis through the centre which makes an angle α with the plane of the circle. If a smooth bead slide on the wire, show that the equation of motion of the bead along the wire is

$$\frac{d^2s}{dt^2} = a\omega^2 \cos^2\alpha \cos\frac{s}{a} \sin\frac{s}{a} - g\cos\alpha \sin\frac{s}{a},$$

where s is measured from the lowest point. Hence find the position of equilibrium of the bead, and the time of a small oscillation about that position."

To integrate this equation, we must first multiply both sides by $\frac{ds}{dt}$; and then integrating with respect to t,

$$\tfrac{1}{2}\left(\frac{ds}{dt}\right)^2 = C - \tfrac{1}{2}a^2\omega^2 \cos^2\alpha \cos^2\frac{s}{a} + ga\cos\alpha\cos\frac{s}{a},$$

or
$$\left(\frac{ds}{dt}\right)^2 = u^2 - a^2\omega^2\cos^2\alpha\left(\cos\frac{s}{a} - \frac{g}{a\omega^2\cos\alpha}\right)^2.$$

Consequently in the position of stable equilibrium of the bead, $\cos\frac{s}{a} = \frac{g}{a\omega^2\cos\alpha}$ ($= \cos\gamma$, suppose) and u is the velocity of the bead relatively to the wire in this position of equilibrium.

Therefore
$$\left(\frac{ds}{dt}\right)^2 = u^2 - a^2\omega^2\cos^2\alpha\left(\cos\frac{s}{a} - \cos\gamma\right)^2.$$

I. If
$$u < a\omega\cos\alpha\,(1-\cos\gamma),$$

the bead will oscillate about the position of stable equilibrium on one side of the lowest point of the wire.

II. If $a\omega\cos\alpha(1-\cos\gamma) < u < a\omega\cos\alpha(1+\cos\gamma)$,

the bead will oscillate through the lowest point of the wire, a position of unstable equilibrium, and beyond the two positions of stable equilibrium on each side of the lowest point.

III. If $u > a\omega\cos\alpha(1+\cos\gamma)$,

the bead will perform complete revolutions of the wire.

To integrate the equation of motion again, put

$$\tan\tfrac{1}{2}\frac{s}{a} = z, \quad \text{or } s = 2a\tan^{-1}z;$$

therefore

$$\frac{ds}{dt} = \frac{2a}{1+z^2}\frac{dz}{dt},$$

and

$$\frac{4a^2}{(1+z^2)^2}\left(\frac{dz}{dt}\right)^2 = u^2 - a^2\omega^2\cos^2\alpha\left(\frac{1-z^2}{1+z^2} - \cos\gamma\right)^2,$$

or

$$4a^2\left(\frac{dz}{dt}\right)^2 = u^2(1+z^2)^2 - a^2\omega^2\cos^2\alpha(1-\cos\gamma-1+\cos\gamma\, z^2)^2$$

$$= \{u - a\omega\cos\alpha.1 - \cos\gamma + z^2(u + a\omega\cos\alpha.1 + \cos\gamma)\},$$

$$\{u + a\omega\cos\alpha.1 - \cos\gamma + z^2(u - a\omega\cos\alpha.1 + \cos\gamma)\},$$

and therefore z is an elliptic function of t.

I. Let $u < a\omega\cos\alpha(1-\cos\gamma)$,

and let s_1, s_2 be the extreme values of s, then

$$\tan^2\tfrac{1}{2}\frac{s_1}{a} = \frac{u + a\omega\cos\alpha(1-\cos\gamma)}{a\omega\cos\alpha(1+\cos\gamma) - u},$$

$$\tan^2\tfrac{1}{2}\frac{s_2}{a} = \frac{a\omega\cos\alpha(1-\cos\gamma) - u}{u + a\omega\cos\alpha(1+\cos\gamma)},$$

or

$$\cos\frac{s_1}{a} = \cos\gamma - \frac{u}{a\omega\cos\alpha},$$

$$\cos\frac{s_2}{a} = \cos\gamma + \frac{u}{a\omega\cos\alpha};$$

and, therefore,

$$\cos\frac{s_1}{a} + \cos\frac{s_2}{a} = 2\cos\gamma.$$

Then

$$4a^2\left(\frac{dz}{dt}\right)^2=\{a^2\omega^2\cos^2\alpha(1+\cos\gamma)^2-u^2\}\left(\tan^2\tfrac{1}{2}\frac{s_1}{a}-z^2\right)\left(z^2-\tan^2\tfrac{1}{2}\frac{s_2}{a}\right)$$

$$=d^2\omega^2\cos^2\alpha\left(1+\cos\frac{s_1}{a}\right)\left(1+\cos\frac{s_2}{a}\right)\left(\tan^2\tfrac{1}{2}\frac{s_1}{a}-z^2\right)\left(z^2-\tan^2\tfrac{1}{2}\frac{s_2}{a}\right),$$

$$\text{or}\quad\left(\frac{dz}{dt}\right)^2=\omega^2\cos^2\alpha\cos^2\tfrac{1}{2}\frac{s_1}{a}\cos^2\tfrac{1}{2}\frac{s_2}{a}\left(\tan^2\tfrac{1}{2}\frac{s_1}{a}-z^2\right)\left(z^2-\tan^2\tfrac{1}{2}\frac{s_2}{a}\right);$$

and therefore

$$\tan\tfrac{1}{2}\frac{s}{a}=z=\tan\tfrac{1}{2}\frac{s_1}{a}\,\mathrm{dn}\,K\frac{t}{T},$$

when
$$k'^2=1-k^2=\frac{\tan^2\tfrac{1}{2}\frac{s_2}{a}}{\tan^2\tfrac{1}{2}\frac{s_1}{a}},$$

and
$$\frac{K}{T}=\omega\cos\alpha\sin\tfrac{1}{2}\frac{s_1}{a}\cos\tfrac{1}{2}\frac{s_2}{a}.$$

The time of oscillation is then $2T$.

When the oscillation is small we must put $s_1=s_2$, and then $k'=1$, $k=0$, $K=\frac{1}{2}\pi$, and

$$2T=\frac{\pi}{\omega\cos\alpha\sin\frac{1}{2}\gamma\cos\frac{1}{2}\gamma},$$

or
$$T=\frac{\pi}{\omega\cos\alpha\sin\gamma}.$$

II. If $a\omega\cos\alpha(1-\cos\gamma)<u<a\omega\cos\alpha(1+\cos\gamma)$,

denoting by s_1 the extreme value of s, we have

$$4a^2\left(\frac{dz}{dt}\right)^2+\{a^2\omega^2\cos^2\alpha(1+\cos\gamma)^2-u^2\}\left(\tan^2\tfrac{1}{2}\frac{s_1}{a}-z^2\right)(z^2+c^2),$$

where, as before,

$$\cos\frac{s_1}{a}=\cos\gamma-\frac{u}{a\omega\cos\alpha},$$

and
$$c^2=\frac{u-a\omega\cos\alpha(1-\cos\gamma)}{u+a\omega\cos\alpha(1+\cos\gamma)},$$

and therefore

$$\frac{1+c^2}{1-c^2} = \cos\gamma + \frac{u}{a\omega\cos\alpha}.$$

Therefore, as before,

$$\left(\frac{dz}{dt}\right)^2 = \frac{\omega^2\cos^2\alpha\cos^2\frac{1}{2}\frac{s_1}{a}}{1-c^2}\left(\tan^2\tfrac{1}{2}\frac{s_1}{a} - z^2\right)(z^2+c^2),$$

and therefore

$$\tan\tfrac{1}{2}\frac{s}{a} = z = \tan\tfrac{1}{2}\frac{s_1}{a}\,\mathrm{cn}\,K\frac{t}{T},$$

where

$$\frac{k^2}{k'^2} = \frac{k^2}{1-k^2} = \frac{\tan^2\frac{1}{2}\frac{s_1}{a}}{c^2},$$

or

$$k^2 = \frac{\tan^2\frac{1}{2}\frac{s_1}{a}}{\tan^2\frac{1}{2}\frac{s_1}{a} + c^2},$$

and

$$\frac{K^2}{T^2} = \omega^2\cos^2\alpha\,\frac{\tan^2\frac{1}{2}\frac{s_1}{a} + c^2}{1-c^2},$$

and the time of oscillation is $2T$, the bead now oscillating through the lowest point of the wire.

III. If $u > a\omega\cos\alpha(1+\cos\gamma)$, the bead will perform complete revolutions, and

$$4a^2\left(\frac{dz}{dt}\right)^2 = \{u^2 - a^2\omega^2\cos^2\alpha(1+\cos\gamma)^2\}(z^2+b^2)(z^2+c^2),$$

when

$$b^2 = \frac{u + a\omega\cos\alpha(1-\cos\gamma)}{u - a\omega\cos\alpha(1+\cos\gamma)},$$

and

$$c^2 = \frac{u - a\omega\cos\alpha(1-\cos\gamma)}{u + a\omega\cos\alpha(1+\cos\gamma)},$$ as before.

$$\frac{b^2+1}{b^2-1} = \cos\gamma - \frac{u}{a\omega\cos\alpha},$$

$$\frac{1+c^2}{1-c^2} = \cos\gamma + \frac{u}{a\omega\cos\alpha}.$$

Therefore $$\left(\frac{dz}{dt}\right)^2 = \frac{b^2\omega^2\cos^2\alpha}{(1-b^2)(1-c^2)}(z^2+b^2)(z^2+c^2),$$

and therefore $$\tan\tfrac{1}{2}\frac{s}{a} = z = b\,\mathrm{tn}\,K\frac{t}{T},$$

where $$k'^2 = 1-k^2 = \frac{b^2}{c^2},$$

and $$\frac{K^2}{T^2} = \frac{b^2c^2\omega^2\cos^2\alpha}{(1-b^2)(1-c^2)},$$

and the time of a complete revolution is $2T$.

NOTE ON A THEOREM OF PROF. CAYLEY'S.

By *R. F. Scott, M.A.*

At page 164, vol. v. of the *Messenger*, Prof. Cayley gave the following: "Theorem in Trigonometry.

If $$A+B+C+F+G+H=0,$$

then $$\begin{vmatrix} \sin\overline{A+F}\,\sin\overline{B+F}\,\sin\overline{C+F}, & \cos F, & \sin F \\ \sin\overline{A+G}\,\sin\overline{B+G}\,\sin\overline{C+G}, & \cos G, & \sin G \\ \sin\overline{A+H}\,\sin\overline{B+H}\,\sin\overline{C+H}, & \cos H, & \sin H \end{vmatrix} = 0."$$

In seeking to establish this theorem the following method occurred to me, which as it is applicable to all similar determinants, is not without interest.

If we write $a = e^{iA}$, &c., the first row of the determinant is

$$\frac{1}{2^3i^3}\left(af-\frac{1}{af}\right)\left(bf-\frac{1}{bf}\right)\left(cf-\frac{1}{cf}\right),\ \frac{1}{2}\left(f+\frac{1}{f}\right),\ \frac{1}{2i}\left(f-\frac{1}{f}\right)$$ similic

Hence we can take the factor $\frac{1}{2^5i^3}$ outside the deter

Writing only the first row, the others being got by changes, the determinant is now

$$D = \frac{1}{2^5}\left|\ \frac{(a^2f^2-1)(b^2f^2-1)(c^2f^2-1)}{abcf^3}, \frac{f^2+1}{f}, \frac{f^2-1}{f}+B+C\right).$$

$$= \frac{1}{2^5abcf^3g^3h^3}\left|\ a^2b^2c^2f^6-(a^2b^2+b^2c^2+c^2a^2)f^4 + (a^2+b^2+c^2)f^2-1,\ f^4+f^2\right.$$

Now add the second column to the last and divide by 2, then subtract the last column from the second, thus

$$D = \frac{1}{2^4 abcf^3g^3h^3} \mid a^2b^2c^2f^6 - (a^2b^2 + b^2c^2 + c^2a^2)f^4 + (a^2 + b^2 + c^2)f^2 - 1,\ f^2,\ f^4 \mid .$$

Now multiply the last column by $a^2b^2 + b^2c^2 + c^2a^2$ and add it to the first, then the second column by $a^2 + b^2 + c^2$ and subtract it from the first, thus

$$D = \frac{1}{2^4 abcf^3g^3h^3} \mid a^2b^2c^2f^6 - 1,\ f^2,\ f^4 \mid$$

$$= \frac{1}{2^4 abcf^3g^3h^3} \{a^2b^2c^2f^2g^2h^2 \mid f^4, 1, f^2 \mid - \mid 1, f^2, f^4 \mid\}$$

$$= \frac{1}{2^4 abcf^3g^3h^3} \{a^2b^2c^2f^2g^2h^2 - 1\}(g^2 - h^2)(h^2 - f^2)(f^2 - g^2)$$

$$= \frac{1}{2^4} \cdot \frac{g^2 - h^2}{gh} \cdot \frac{h^2 - f^2}{hf} \cdot \frac{f^2 - g^2}{fg} \left\{abcfgh - \frac{1}{abcfgh}\right\}$$

$$= \sin(C - H)\sin(H - F)\sin(F - G)\sin(A + B + C + F + G + H).$$

This method will be found to lead with great facility to the evaluation of all determinants whose elements are sines and cosines.

The following examples may be given:

The value of the determinant

$$4a^2 \left(\frac{}{dt}\right), \quad \begin{vmatrix} 1, & 1, & 1, & 1 \\ \cos A, & \cos B, & \cos C, & \cos D \\ \sin A, & \sin B, & \sin C, & \sin D \\ \sin 3A, & \sin 3B, & \sin 3C, & \sin 3D \end{vmatrix},$$

when

and $$\Pi \sin\tfrac{1}{2}(A - B)\{\cos(S + A) + \cos(S + B) + \cos(S + C) + \cos(S + D)\},$$

$= A + B + C + D$, and in the product of the sines of fferences we adhere to the order of the alphabet, subtract any letter from one which follows it.

Similarly

$$\begin{vmatrix} \sin A, & \sin B, & \sin C, & \sin D \\ \cos A, & \cos B, & \cos C, & \cos D \\ \sin 2A, & \sin 2B, & \sin 2C, & \sin 2D \\ \cos 2A, & \cos 2B, & \cos 2C, & \cos 2D \end{vmatrix}$$

$$= 2^5 \Pi \sin \tfrac{1}{2}(A-B) \{\cos \tfrac{1}{2}(A+B-C-D) + \cos \tfrac{1}{2}(A+C-B-D) + \cos \tfrac{1}{2}(A+D-B-C)\}.$$

We have also the general theorem for $2n+1$ angles

$$\begin{vmatrix} 1 & 1 & & 1 \\ \sin a_1, & \sin a_2, & \ldots & \sin a_{2n+1} \\ \cos a_1, & \ldots\ldots & & \ldots\ldots \\ \sin 2a_1, & \ldots\ldots & & \ldots\ldots \\ \cos 2a_1, & \ldots\ldots & & \ldots\ldots \\ \vdots & & & \\ \sin na_1, & \ldots\ldots & & \ldots\ldots \\ \cos na_1, & \ldots\ldots & & \ldots\ldots \end{vmatrix} = 2^{n^2} \Pi \sin \tfrac{1}{2}(a_i - a_k) \; (i > k),$$

and the following curious theorem:

The determinant

$$D = \begin{vmatrix} \cos a & , \cos(a+b) & , \cos(a+2b) \ldots & \cos\{a+(n-1)b\} \\ \cos(a+b) & , \cos(a+2b), & \ldots\ldots\ldots\ldots & \cos(a+nb) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \cos\{a+(n-1)b\}, & \ldots\ldots & \ldots\ldots & \cos\{a+(2n-1)b\} \end{vmatrix}$$

vanishes for all values of n greater than 2. When $n=2$ its value is $-\sin^2 b$. The method applies equally to hyperbolic and to some of the gudermannian functions, *e.g.*

$$\begin{vmatrix} \cosh A, & \cosh B, & \cosh C \\ \sinh A, & \sinh B, & \sinh C \\ \cosh 3A, & \cosh 3B, & \cosh 3C \end{vmatrix}$$

$$= 4 \sinh(B-C) \sinh(C-A) \sinh(A-B) \cosh(A+B+C).$$

ON A CLASS OF DETERMINANTS, WITH A NOTE ON PARTITIONS.

By *J. W. L. Glaisher.*

§ 25. The following remarks may be regarded as a continuation of two previous papers, "Expressions for Laplace's coefficients, Bernoullian and Eulerian numbers, &c. as determinants" (vol. VI. pp. 49–63), which contains §§ 1–13, and "On a class of determinants" (vol. VII. pp. 160–165) which contains §§ 14–24.

§ 26. It may be observed that the determinant

$$\begin{vmatrix} a_1-b_1, & 1, & ., & ., \dots \\ a_2-b_2, & a_1, & 1, & ., \dots \\ a_3-b_3, & a_2, & a_1, & 1, \dots \\ a_4-b_4, & a_3, & a_2, & a_1, \dots \\ \dots\dots\dots\dots\dots\dots \end{vmatrix},$$

which is specially considered in the second paper, may be written in a slightly different form; for

$$\begin{vmatrix} a_1-b_1, & 1, & ., & ., \dots \\ a_2-b_2, & a_1, & 1, & ., \dots \\ a_3-b_3, & a_2, & a_1, & 1, \dots \\ a_4-b_4, & a_3, & a_2, & a_1, \dots \\ \dots\dots\dots\dots\dots\dots \end{vmatrix} (n \text{ rows}) = \begin{vmatrix} 1, & 1, & ., & ., \dots \\ b_1, & a_1, & 1, & ., \dots \\ b_2, & a_2, & a_1, & 1, \dots \\ b_3, & a_3, & a_2, & a_1, \dots \\ \dots\dots\dots\dots\dots\dots \end{vmatrix} (n+1 \text{ rows})$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1).$$

This is at once seen to be true by subtracting the second column from the first in the second determinant.

We can prove directly that, if

$$\frac{1+b_1x+b_2x^2+b_3x^3+\&c}{1+a_1x+a_2x^2+a_3x^3+\&c.} = 1-P_1x+P_2x^2-P_3x^3+\&c.,$$

then

$$P_n = \begin{vmatrix} 1, & 1, & ., & ., \dots \\ b_1, & a_1, & 1, & ., \dots \\ b_2, & a_2, & a_1, & 1, \dots \\ b_3, & a_3, & a_2, & a_1, \dots \\ \dots\dots\dots\dots\dots\dots \end{vmatrix} (n+1 \text{ rows}),$$

for this value of P_n follows immediately from the system of equations

$$\begin{aligned} 1 &= 1, \\ -P_1 + a_1 &= b_1, \\ P_2 - a_1 P_1 + a_2 &= b_2, \\ -P_3 + a_1 P_2 - a_2 P_1 + a_3 &= b_3, \\ \&c. &= \&c. \end{aligned}$$

obtained by equating the coefficients of the powers of x.

§ 27. *Theorem.* If

$$\log(1 + a_1 x + a_2 x^2 + a_3 x^3 + \&c.) = P_1 x - \tfrac{1}{2} P_2 x^2 + \tfrac{1}{3} P_3 x^3 - \&c.,$$

then
$$P_n = \begin{vmatrix} a_1, & 1, & ., & ., \ldots \\ 2a_2, & a_1, & 1, & ., \ldots \\ 3a_3, & a_2, & a_1, & 1, \ldots \\ 4a_4, & a_3, & a_2, & a_1, \ldots \\ \ldots\ldots & & & \end{vmatrix} (n \text{ rows}).$$

Proof. By differentiation

$$\frac{a_1 + 2a_2 x + 3a_3 x^2 + \&c.}{1 + a_1 x + a_2 x^2 + a_3 x^3 + \&c.} = P_1 - P_2 x + P_3 x^2 - \&c.,$$

whence, by § 26,

$$\frac{P_n}{a_1} = \begin{vmatrix} 1, & 1, & ., & \ldots \\ \dfrac{2a_2}{a_1}, & a_1, & 1, & \ldots \\ \dfrac{3a_3}{a_1}, & a_2, & a_1, & \ldots \\ \ldots\ldots & & & \end{vmatrix} (n \text{ rows});$$

and, therefore P_n is as above.

§ 28. *Theorem.* If

$$\exp(a_1 x + a_2 x^2 + a_3 x^3 + \&c.) = 1 + P_1 x + P_2 x^2 + P_3 x^3 + \&c.,$$

then
$$P_n = \frac{1}{n!} \begin{vmatrix} a_1, & -1, & ., & ., \ldots \\ 2a_2, & a_1, & -2, & ., \ldots \\ 3a_3, & 2a_2, & a_1, & -3, \ldots \\ 4a_4, & 3a_3, & 2a_2, & a_1, \ldots \\ \ldots\ldots & & & \end{vmatrix} (n \text{ rows}).$$

Proof. By differentiation and division

$$a_1 + 2a_2x + 3a_3x^2 + \&c. = \frac{P_1 + 2P_2x + 3P_3x^2 + \&c.}{1 + P_1x + P_2x^2 + P_3x^3 + \&c.};$$

whence, equating coefficients,

$$\begin{aligned}
a_1 &= P_1,\\
2a_2 + a_1P_1 &= 2P_2,\\
3a_3 + 2a_2P_1 + a_1P_2 &= 3P_3,\\
4a_4 + 3a_3P_1 + 2a_2P_2 + a_1P_3 &= 4P_4,\\
\&c. &= \&c.,
\end{aligned}$$

therefore

$$\begin{aligned}
P_1 &= -a_1,\\
a_1P_1 - 2P_2 &= -2a_2,\\
2a_2P_1 + a_1P_2 - 3P_3 &= -3a_3,\\
3a_3P_1 + 2a_2P_2 + a_1P_3 - 4P_4 &= -4a_4,\\
\&c. &= \&c.,
\end{aligned}$$

giving the above value of P_n.

If the left-hand side be $\exp(a_0 + a_1x + a_2x^2 + \&c.)$ instead of $\exp(a_1x + a_2x^2 + \&c.)$, it is only necessary to multiply throughout by $\exp a_0$.

As an example, let $a_1x + a_2x^2 + a_3x^3 + \&c. = e^x$, then, by the theorem,

$$\exp e^x = 1 + P_1x + P_2x^2 + P_3x^3 + \&c.,$$

where
$$P_n = \frac{e}{n!}\begin{vmatrix} 1, & -1, & ., & ., & \cdots \\ 1, & 1, & -2, & ., & \cdots \\ \frac{1}{2!}, & 1, & 1, & -3, & \cdots \\ \frac{1}{3!}, & \frac{1}{2!}, & 1, & 1, & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} (n \text{ rows}),$$

agreeing with the result in § 12.

§ 29. It results from the theorems in the last two sections that if

$$b_n = (-)^{n-1}\frac{1}{n}\begin{vmatrix} a_1, & 1, & ., & ., \ldots \\ 2a_2, & a_1, & 1, & ., \ldots \\ 3a_3, & a_2, & a_1, & 1, \ldots \\ 4a_4, & a_3, & a_2, & a_1, \ldots \\ \ldots\ldots & & & \end{vmatrix} \quad (n \text{ rows}),$$

then

$$a_n = \frac{1}{n!}\begin{vmatrix} b_1, & -1, & ., & ., \ldots \\ 2b_2, & b_1, & -2, & ., \ldots \\ 3b_3, & 2b_2, & b_1, & -3, \ldots \\ 4b_4, & 3b_3, & 2b_2, & b_1, \ldots \\ \ldots\ldots & & & \end{vmatrix} \quad (n \text{ rows}),$$

and *vice versâ*.

For example, if

$$b_1 = a_1,$$
$$b_2 = \tfrac{1}{2}(2a_2 - a_1^2),$$
$$b_3 = \tfrac{1}{3}(a_1^3 - 3a_1a_2 + 3a_3),$$

and the theorem asserts that

$$a_3 = \tfrac{1}{6}\begin{vmatrix} a_1 & , & -1 & , & . \\ 2a_2 - a_1^2 & , & a_1 & , & -2 \\ a_1^3 - 3a_1a_2 + 3a_3, & & 2a_2 - a_1^2, & & a_1 \end{vmatrix}.$$

Adding the second column, multiplied by a_1, to the first column, the right-hand side is at once seen to be equal to a_3.

§ 30. In § 27 let $a_n = n$, and the theorem shows that

$$\log(1 + x + 2x^2 + 3x^3 + \&c.) = P_1x - \tfrac{1}{2}P_2x^2 + \tfrac{1}{3}P_3x^3 - \&c.,$$

where

$$P_n = \begin{vmatrix} 1^2, & 1, & ., & ., \ldots \\ 2^2, & 1, & 1, & ., \ldots \\ 3^2, & 2, & 1, & 1, \ldots \\ 4^2, & 3, & 2, & 1, \ldots \\ \ldots\ldots & & & \end{vmatrix} \quad (n \text{ rows}).$$

Now

$$\log(1 + x + 2x^2 + 3x^3 + \&c.) = \log\left(1 + \frac{x}{(1-x)^2}\right)$$
$$= \log\frac{1 - x + x^2}{(1-x)^2} = \log\frac{1 + x^3}{(1-x)(1-x^3)}$$
$$- \log(1-x) - \log(1-x^3) + \log(1+x^3)$$
$$= x + \tfrac{1}{2}x^2 + \&c. + x^3 + \tfrac{1}{2}x^4 + \&c. + x^3 - \tfrac{1}{2}x^6 + \&c.,$$

and the coefficient of $\frac{1}{n}x^n = 0, 1, 3, 4, 3$ or 1, according as

$$n \equiv 0, 1, 2, 3, 4 \text{ or } 5 \pmod 6.$$

It therefore follows that the determinant

$$\left|\begin{array}{l} 1^2,\ 1,\ .,\ .,\ \ldots \\ 2^2,\ 1,\ 1,\ .,\ \ldots \\ 3^2,\ 2,\ 1,\ 1,\ \ldots \\ 4^2,\ 3,\ 2,\ 1,\ \ldots \\ \ldots\ldots\ldots\ldots \end{array}\right| \quad (n \text{ rows})$$

$= 0, 1, -3, 4, -3$ or 1, according as

$$n \equiv 0, 1, 2, 3, 4 \text{ or } 5 \pmod 6.$$

§ 31. This can be proved directly as follows. Subtracting the second column from the first, the determinant becomes

$$\left|\begin{array}{l} .,\ 1,\ .,\ .,\ \ldots \\ 3,\ 1,\ 1,\ .,\ \ldots \\ 7,\ 2,\ 1,\ 1,\ \ldots \\ 13,\ 3,\ 2,\ 1,\ \ldots \\ \ldots\ldots\ldots\ldots \end{array}\right| \quad (n \text{ rows})$$

$$= -\left|\begin{array}{l} 3,\ 1,\ .,\ .,\ \ldots \\ 7,\ 1,\ 1,\ .,\ \ldots \\ 13,\ 2,\ 1,\ 1,\ \ldots \\ 21,\ 3,\ 2,\ 1,\ \ldots \\ \ldots\ldots\ldots\ldots \end{array}\right| \quad (n-1 \text{ rows}).$$

Subtract the second column, multiplied by 3, from the first column, and the determinant

$$= -\left|\begin{array}{l} .,\ 1,\ .,\ .,\ \ldots \\ 4,\ 1,\ 1,\ .,\ \ldots \\ 7,\ 2,\ 1,\ 1,\ \ldots \\ 12,\ 3,\ 2,\ 1,\ \ldots \\ \ldots\ldots\ldots\ldots \end{array}\right| \quad (n-1 \text{ rows})$$

$$= \left|\begin{array}{l} 4,\ 1,\ .,\ .,\ \ldots \\ 7,\ 1,\ 1,\ .,\ \ldots \\ 12,\ 2,\ 1,\ 1,\ \ldots \\ 19,\ 3,\ 2,\ 1,\ \ldots \\ \ldots\ldots\ldots\ldots \end{array}\right| \quad (n-2 \text{ rows})$$

Subtract the second column multiplied by 4, and this

$$= - \begin{vmatrix} 3, & 1, & ., & ., \ldots \\ 4, & 1, & 1, & ., \ldots \\ 7, & 2, & 1, & 1, \ldots \\ 12, & 3, & 2, & 1, \ldots \\ \ldots\ldots\ldots\ldots\ldots \end{vmatrix} \quad (n-3 \text{ rows}),$$

and so on.

The changes in the first column are exhibited below:

1,	4,	9,	16,	25,	36,	49,	64,	81,	100,	121,	144, 169, ...
1,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11, 12, ...
0,	3,	7,	13,	21,	31,	43,	57,	73,	91,	111,	133, 157, ...
	3,	3,	6,	9,	12,	15,	18,	21,	24,	27,	30, 33, ...
	0,	4,	7,	12,	19,	28,	39,	52,	67,	84,	103, 124, ...
		4,	4,	8,	12,	16,	20,	24,	28,	32,	36, 40, ...
		0,	3,	4,	7,	12,	19,	28,	39,	52,	67, 84, ...
			3,	3,	6,	9,	12,	15,	18,	21,	24, 27, ...
			0,	1,	1,	3,	7,	13,	21,	31,	43, 57, ...
				1,	1,	2,	3,	4,	5,	6,	7, 8, ...
				0,	0,	1,	4,	9,	16,	25,	36, 49, ...

Thus, the leading elements of the determinants are 1, 3, 4, 3, 1, 0, and then the determinant becomes the same as the original determinant but reduced by six rows. This is easily seen to be true generally, for from n^2 we subtract successively $n-1$, $3n-6$, $4n-12$, $3n-12$, $n-5$, that is, $12n-36$; so that after the subtractions n^2 becomes $(n-6)^2$, and at the same time the determinant is reduced by six rows.

§ 32. Similarly, since

$$\log(1+2x+3x^2+4x^3+\&c.) = -2\log(1-x)$$
$$= 2(x+\tfrac{1}{2}x^2+\tfrac{1}{3}x^3+\&c).,$$

we have

$$\begin{vmatrix} 1, & 1, & ., & ., \ldots \\ 3, & 2, & 1, & ., \ldots \\ 6, & 3, & 2, & 1, \ldots \\ 10, & 4, & 3, & 2, \ldots \\ \ldots\ldots\ldots\ldots\ldots \end{vmatrix} \quad (n \text{ rows}),$$

(the first column consisting of the triangular numbers) $=-1$ or 1 according as n is even or uneven. This can also be proved at once by the method employed in the last section.

As another example, we see from the equation

$$\log(1+2^2x+3^2x^2+4^2x^3+\&c.)=\log\frac{1+x}{(1-x)^3},$$

that

$$\begin{vmatrix} 1.2^2, & 1^2, & ., & ., \ldots \\ 2.3^2, & 2^2, & 1^2, & ., \ldots \\ 3.4^2, & 3^2, & 2^2, & 1^2, \ldots \\ 4.5^2, & 4^2, & 3^2, & 2^2, \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix} \quad (n \text{ rows})$$

$=-2$ or 4 according as n is even or uneven.

Writing the first column of this determinant in the form 2^3-2^2, 3^3-3^2, &c., it is evident by transforming the determinant as in § 26, that, if $n>1$,

$$\begin{vmatrix} 1^3, & 1^2, & ., & ., \ldots \\ 2^3, & 2^2, & 1^2, & ., \ldots \\ 3^3, & 3^2, & 2^2, & 1^2, \ldots \\ 4^3, & 4^2, & 3^2, & 2^2, \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix} \quad (n \text{ rows})$$

$=-4$ or 2 according as n is even or uneven. This result also follows at once from the latter part of § 26; for

$$\frac{1+2^3x+3^3x^2+4^3x^3+\&c.}{1+2^2x+3^2x^2+4^2x^3+\&c.}=\frac{1+4x+x^2}{1-x^2}$$
$$=1+4x+2x^2+4x^3+2x^4+\&c.$$

§§ 33—38. *Partitions.*

§ 33. The total number of partitions of any number n, order being attended to, is 2^{n-1}; and if the partitions containing an uneven number of parts be treated as negative the number of partitions is zero. By the words "order being attended to" it is meant that, for example, $3+4$ and $4+3$, or $1+1+5$, $1+5+1$ and $5+1+1$ are to be regarded as different partitions, so that if a partition of n which contains r parts has a parts each equal to α, b parts each equal to β, c parts each equal to γ, &c., then this partition is to be counted $\frac{r!}{a!\,b!\,c!\ldots}$ times, and therefore

$$\Sigma\frac{r!}{a!\,b!\,c!\ldots}=2^{n-1},\quad \Sigma(-)^r\frac{r!}{a!\,b!\,c!\ldots}=0,$$

the Σ's referring to all the partitions of n.

Take, for example, the partitions of 5; these are

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,$$
$$2+1+1+1,\ 1+1+1+1+1,$$

and therefore

$$1+(2!+2!)+\left(\frac{3!}{2!}+\frac{3!}{2!}\right)+\frac{4!}{3!}+\frac{5!}{5!}=2^4,$$

$$1-(2!+2!)+\left(\frac{3!}{2!}+\frac{3!}{2!}\right)-\frac{4!}{3!}+\frac{5!}{5!}=0,$$

These theorems, which result from the equations

$$\frac{1}{1-(x+x^2+x^3+\&c.)}=\frac{1-x}{1-2x},$$

$$\frac{1}{1+(x+x^2+x^3+\&c.)}=1-x,$$

are well-known (see Sylvester, *British Association Report*, Edinburgh, 1871, *Transactions of the Sections*, p. 24).

§ 34. In the same manner from the equations

$$\log\frac{1}{1-(x+x^2+x^3+\&c.)}=\log(1-x)-\log(1-2x),$$

$$\log\frac{1}{1+(x+x^2+x^3+\&c.)}=\log(1-x),$$

we see that

$$\Sigma\frac{1}{r}\frac{r!}{a!\ b!\ c!\ldots}=\frac{2^n-1}{n},\quad \Sigma(-)^{r-1}\frac{1}{r}\frac{r!}{a!\ b!\ c!\ldots}=\frac{1}{n}.$$

For example, if $n=5$ these give

$$1+\tfrac{1}{2}(2!+2!)+\tfrac{1}{3}\left(\frac{3!}{2!}+\frac{3!}{2!}\right)+\tfrac{1}{4}\frac{4!}{3!}+\tfrac{1}{5}\frac{5!}{5!}=\frac{2^5-1}{5},$$

$$1-\tfrac{1}{2}(2!+2!)+\tfrac{1}{3}\left(\frac{3!}{2!}+\frac{3!}{2!}\right)-\tfrac{1}{4}\frac{4!}{3!}+\tfrac{1}{5}\frac{5!}{5!}=\tfrac{1}{5}.$$

If we omit the partition, which consists entirely of 1's, the second equation, in the case of n even, becomes

$$\Sigma(-)^r\frac{1}{r}\frac{r!}{a!\ b!\ c!\ldots}=0,$$

§ 35. From the equation in § 30, viz.

$$\log(1+x+2x^2+3x^3+\&c.) = -\log(1-x) - \log(1-x^2) + \log(1+x^3),$$

it follows that if p denotes the product of the parts, supposed r in number, of a partition of n, then, if the order be attended to,

$$\Sigma(-)^{r-1}\frac{p}{r} = 0, \frac{1}{n}, \frac{3}{n}, \frac{4}{n}, \frac{3}{n} \text{ or } \frac{1}{n} \ldots\ldots\ldots \text{(i.)},$$

according as $n \equiv 0, 1, 2, 3, 4$, or $5 \pmod 6$.

If we consider only the different partitions (without regard to the order of the parts in each) then in (i.) we are to replace

$$\Sigma(-)^{r-1}\frac{p}{r} \text{ by } \Sigma(-)^{r-1}\frac{1}{r}\frac{r!}{a!\,b!\,c!\ldots}p.$$

For example, if $n = 5$ the formula gives

$$5 - \tfrac{1}{2}(2!\,4 + 2!\,6) + \tfrac{1}{3}\left(\frac{3!}{2!}3 + \frac{3!}{2!}4\right) - \tfrac{1}{4}\frac{4!}{3!}2 + \tfrac{1}{5}\frac{5!}{5!} = \tfrac{1}{5},$$

viz. $$5 - 10 + 7 - 2 + \tfrac{1}{5} = \tfrac{1}{5}.$$

§ 36. By § 27,

$$\log(1 + x + 2^m x^2 + 3^m x^3 + \&c.) = P_1 x - \tfrac{1}{2}P_2 x^2 + \tfrac{1}{3}P_3 x^3 - \&c.,$$

where $$P_n = \begin{vmatrix} 1^{m+1}, & 1, & ., & ., \ldots \\ 2^{m+1}, & 1^m, & 1, & ., \ldots \\ 3^{m+1}, & 2^m, & 1^m, & 1, \ldots \\ 4^{m+1}, & 3^m, & 2^m, & 1^m, \ldots \\ \ldots\ldots\ldots & & & \end{vmatrix} \quad (n \text{ rows}),$$

and therefore this determinant

$$= (-)^n n \Sigma(-)^r \frac{1}{r}\frac{r!}{a!\,b!\,c!\ldots}p^m.$$

For example, put $n = 4$; since the partitions of 4 are $4, 3+1, 2+2, 2+1+1, 1+1+1+1$, this formula gives

$$\begin{vmatrix} 1^{m+1}, & 1, & ., & . \\ 2^{m+1}, & 1^m, & 1, & . \\ 3^{m+1}, & 2^m, & 1^m, & 1 \\ 4^{m+1}, & 3^m, & 2^m, & 1^m \end{vmatrix} = 4\{-4^m + \tfrac{1}{2}(2.3^m + 4^m) - \tfrac{1}{3}.3.2^m + \tfrac{1}{4}.1^m\},$$

which is readily verified.

§ 37. The formula in the last section is still true if we substitute $a_1, a_2, a_3 \ldots$ for $1^m, 2, 3^m \ldots$ and $a_1, 2a_2, 3a_3 \ldots$ for $1^{m+1}, 2^{m+1}, 3^{m+1} \ldots$ and if for p^m, a product formed from $1^m, 2^m, 3^m \ldots$ we substitute π, a similar product formed from $a_1, a_2, a_3 \ldots$, viz.

$$\begin{vmatrix} a_1, & 1, & ., & . & \ldots \\ 2a_2, & a_1, & 1, & . & \ldots \\ 3a_3, & a_2, & a_1, & 1 & \ldots \\ 4a_4, & a_3, & a_2, & a_1 & \ldots \\ \ldots\ldots & & & & \end{vmatrix} \quad (n \text{ rows})$$

$$= (-)^n n \Sigma (-)^r \frac{\pi}{r} \text{ or } = (-)^n n \Sigma (-)^r \frac{1}{r} \frac{r!}{a!\, b!\, c! \ldots} \pi,$$

according as the order of the parts in the partition is or is not attended to.

Also we have

$$\begin{vmatrix} a_1, & 1, & ., & ., \ldots \\ a_2, & a_1, & 1, & ., \ldots \\ a_3, & a_2, & a_1, & 1, \ldots \\ a_4, & a_3, & a_2, & a_1, \ldots \\ \ldots\ldots & & & \end{vmatrix} \quad (n \text{ rows})$$

$$= (-)^n \Sigma (-)^r \pi \text{ or } (-)^n \Sigma (-)^r \frac{r!}{a!\, b!\, c! \ldots} \pi,$$

according as the order of the parts in the partitions is or is not attended to.

§ 38. The formulæ in §§ 33—35 would all be useful in verifying that in writing down all the partitions of a number, none had been omitted, but those in §§ 36, 37 would be of no practical value for this purpose on account of the labour of calculating the determinants. It is scarcely necessary to remark that in general the determinant expressions considered in this paper do not afford a good method of obtaining the numerical values of the coefficient which they represent.

ERRATUM.—In § 24 (vol. VII, p. 165, line 3) for "$\log(1+x) = \log x + C_1 x +$ &c." read "$\log\log(1+x) = \log x + C_1 x +$ &c."

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, December 12th, 1878, C. W. Merrifield, Esq., F.R.S., *President*, in the Chair. Prof. W. Stanley Jevons, F.R.S., was elected a Member. The following communications were made: "On a Kinematic Paradox (the Rotameter)," Mr. H. Perigal, F.R.A.S.; "On Forms of Numbers determined by Continued Fractions," Mr. S. Roberts, F.R.S.; "On a Graphic Construction of the Powers of a Linear Substitution," Prince Camille de Polignac.

Thursday, January 9th, 1879, C. W. Merrifield, Esq., F.R.S., *President*, in the Chair. Dr. J. Hopkinson, F.R.S., was admitted into the Society. The following communications were made to the Society: "On a Theorem in Elliptic Functions," Prof. Cayley, F.R.S. (Mr. J. W. L. Glaisher and Prof. Smith spoke on the subject of this paper); "On a New Modular Equation," Prof. H. J. S. Smith, F.R.S.; "On Coefficients of Induction and Capacity of Two Electrified Spheres," Prof. Greenhill; "On Certain Systems of Partial Differential Equations of the First Order with several Dependent Variables," Prof. H. W. Lloyd Tanner.

R. TUCKER, M.A., *Hon. Sec.*

ON THE STRESSES CAUSED IN AN ELASTIC SOLID BY INEQUALITIES OF TEMPERATURE.*

By *J. Hopkinson, F.R.S.*

VARIOUS phenomena due to the stresses caused by inequalities of temperature will occur to everyone. Glass vessels crack when they are suddenly and unequally heated, or when in manufacture they have been allowed to cool so as to be in a state of stress when cold. Optical glass is doubly refracting when badly annealed or when different parts of the mass are at different temperatures. Iron castings which have been withdrawn from the mould whilst still very hot, or of which the form is such that some parts cool more rapidly than others, are liable to break without the application of any considerable external stress. The ordinary theory of elastic solids may easily be applied to some such cases.

Let uvw be the displacements of any point (xyz) of a body density ρ, parallel to the coordinate axes. Let N_1, N_2, N_3, T_1, T_2, T_3 be the elements of stress; *i.e.* $N_1\alpha$ is the tension across an elementary area α resolved parallel to x, the element α being perpendicular to x; $T_1\beta$ is the shearing force across an element β resolved parallel to z, β being perpendicular to y; T_1 is then also the shear parallel to y across an element perpendicular to z.

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* See *Report of the British Association* for 1872, p. 51.

If ρX, ρY, ρZ be the external forces at (xyz)

$$\left.\begin{aligned} \frac{dN_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz} + \rho X = 0 \\ \frac{dT_3}{dx} + \frac{dN_2}{dy} + \frac{dT_1}{dz} + \rho Y = 0 \\ \frac{dT_2}{dx} + \frac{dT_1}{dy} + \frac{dN_3}{dz} + \rho Z = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(1).$$

These are strictly accurate. Of an inferior order of accuracy are the equations expressing the stresses in terms of the strains of an isotropic solid

$$\left.\begin{aligned} N_1 &= \lambda\theta + 2\mu \frac{du}{dx} \\ T_1 &= \mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(2),$$

where $\theta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$ = the dilatation at the point. These equations are inaccurate, inasmuch as they are inapplicable if the strains be not very small, and as even then in all solids which have been examined the stresses depend not only on the then existing strains but in some degree on the strains which the body has suffered in all preceding time (see Boltzmann, *Akad. der Wissensch. zu Wien*, 1874; Kohlrausch, *Pogg. Annalen*, 1876; Thomson, *Proceedings of Royal Society*, 1865; some experiments of my own, *Proceedings of Royal Society*, 1879; Viscosity in Maxwell's *Heat*).

Assuming equations (2) we observe that as these and also (1) are linear, we may superpose the effects of separate causes of stress in a solid when they act simultaneously.

Equations (2) are intended to apply only to cases in which when the stresses vanish the strains vanish, and in which the strains result from stress only and not from inequalities of temperature. The first limitation is easily removed by the principle of superposition. We must determine separately the stresses when no external forces are applied, and then the stresses due to the external forces on the assumption that the solid is unstrained when free and finally add the results. For example, if we are considering a gun or press cylinder, we know that internal pressure will produce the greatest tension in the inner shells, and we can hence at once infer that if the gun or cylinder be so made that normally the inner shells are

in compression the outer in tension it will be stronger to resist internal pressure.

To ascertain the effect of unequal heating, assume that $\lambda\mu$ are independent of the temperature, an assumption of the same order of accuracy as assuming in the theory of conduction of heat that the conductivity is a constant independent of the temperature. Let κ be the coefficient of linear expansion, τ the temperature at any point in excess of a standard temperature. If there be no stresses,

$$\frac{du}{dx} = \frac{dv}{dy} = \frac{dw}{dz} = \kappa\tau;$$

therefore
$$(3\lambda + 2\mu)\,\kappa\tau = \lambda\theta + 2\mu\frac{du}{dx},$$

$$0 = \mu\left(\frac{dw}{dy} + \frac{dv}{dz}\right), \text{ \&c.,}$$

if there were stresses, but τ were zero,

$$N_1 = \lambda\theta + 2\mu\frac{du}{dx}, \text{ \&c.,}$$

superposing effects we have

$$\left.\begin{aligned} N_1 &= \lambda\theta + 2\mu\frac{du}{dx} - (3\lambda + 2\mu)\,\kappa\tau \\ T_1 &= \mu\left(\frac{dw}{dy} + \frac{dv}{dz}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots(3).$$

Substitute in the equations of equilibrium

$$\left.\begin{aligned} (\lambda+\mu)\frac{d\theta}{dx} + \mu\nabla^2 u - \gamma\frac{d\tau}{dx} + \rho X = 0 \\ (\lambda+\mu)\frac{d\theta}{dy} + \mu\nabla^2 v - \gamma\frac{d\tau}{dy} + \rho Y = 0 \\ (\lambda+\mu)\frac{d\theta}{dz} + \mu\nabla^2 w - \gamma\frac{d\tau}{dz} + \rho Z = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots(4),$$

where
$$\gamma = (3\lambda + 2\mu)\,\kappa.$$

If there be equilibrium of temperature $\nabla^2\tau = 0$, *and the effect of unequal heating is exactly the same as that of an*

external, force potential $\frac{(3\lambda + 2\mu)\,\kappa\tau}{\rho}$; in this case we have the equations

$$\left.\begin{aligned} \nabla^2\theta &= \rho\,\frac{d^2\theta}{dt^2} \\ \nabla^2\nabla^2 u &= 0, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

still true and under the same conditions.

Examine the case when there are no bodily forces and when everything is symmetrical about a centre. The displacement at any point is radial, call it U, and the principal stresses are radial and tangential, call them R and T.

The equation of equilibrium is

$$\frac{d(r^2R)}{dr} - 2rT = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and the stresses are expressed by

$$\left.\begin{aligned} R &= \lambda\theta + 2\mu\,\frac{dU}{dr} - \gamma\tau \\ T &= \lambda\theta + 2\mu\,\frac{U}{r} - \gamma\tau \\ \theta &= \frac{1}{r^2}\,\frac{d(r^2U)}{dr} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

substituting

$$(\lambda + 2\mu)\left\{\frac{d^2U}{dr^2} + 2\frac{1}{r}\,\frac{dU}{dr} - 2\,\frac{U}{r^2}\right\} = \gamma\,\frac{d\tau}{dr} \quad \ldots\ldots\ldots\ldots (8);$$

therefore

$$r^2U = \frac{\gamma}{\lambda + 2\mu}\int r^2\tau dr + ar^3 + b \quad \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

where a and b are constants to be determined by a knowledge of R or U for two specified values of r. This equation is of course true whether there be equilibrium of temperature or not.

The interior and exterior surfaces of a homogeneous spherical shell are maintained at different temperatures to find the resulting stresses.

Let r_1, r_2 be the internal and external radii, t_1, t_2 the

internal and external temperatures, then if τ be the temperature at radius r,

$$\left.\begin{aligned} \tau &= c+\frac{f}{r} \\ \text{where}\quad c &= \frac{r_2t_2-r_1t_1}{r_2-r_1} \\ \text{and}\quad f &= -\frac{(t_2-t_1)r_1r_2}{r_2-r_1} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(10).$$

Substitute in equation (9) and then in (7)

$$\left.\begin{aligned} U &= \frac{\gamma}{\lambda+2\mu}\left(\tfrac{1}{3}cr+\tfrac{1}{2}f\right)+ar+\frac{b}{r^2} \\ \theta &= \frac{\gamma}{\lambda+2\mu}\left(c+\frac{f}{r}\right)+3a=\frac{\gamma}{\lambda+2\mu}\tau+3a \\ R &= -\frac{4\mu\gamma}{3(\lambda+2\mu)}c-\frac{2\mu\gamma}{\lambda+2\mu}\frac{f}{r}+(3\lambda+2\mu)a-\frac{4\mu b}{r^3} \end{aligned}\right\} \ldots(11),$$

write R in the form

$$R=A+\frac{B}{r}+\frac{C}{r^3};$$

where $$B=\frac{2\mu\gamma}{\lambda+2\mu}\frac{(t_2-t_1)r_1r_2}{r_2-r_1},$$

for we shall not require to find U; we find

$$A(r_2^3-r_1^3)+B(r_2^2-r_1^2)=0,$$
$$C(r_2^3-r_1^3)+Br_1^2r_2^2(r_2-r_1)=0,$$

whence

$$R=\frac{2\mu\gamma(t_2-t_1)r_1r_2}{r_2^3-r_1^3}\left\{-(r_2+r_1)+\frac{r_2^2+r_1r_2+r_1^2}{r}-\frac{r_1^2r_2^2}{r^3}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

R will have a maximum or minimum value when

$$r^2=\frac{3r_1^2r_2^2}{r_2^2+r_1r_2+r_1^2},$$

and its value then is

$$B'\left\{-(r_2+r_1)+\frac{2(r_2^2+r_1r_2+r_1^2)^{\frac{3}{2}}}{3r_1r_2\sqrt{3}}\right\},$$

this is positive if $t_2>t_1$, as we may see at once from physical reasons.

Now $$T=\frac{1}{2r}\frac{dr^2R}{dr}=B'\left\{-(r_2+r_1)+\frac{r_2^2+r_1r_2+r_1^2}{2r}+\frac{r_1^2r_2^2}{2r^3}\right\},$$

if $t_2 > t_1$, this decreases as r increases, when $r = r_1$, its value becomes

$$B' \left\{ \frac{-2(r_2+r_1)r_1 + (r_2^2 + r_1 r_2 + r_1^2) + r_2^2}{2r_1} \right\}$$

$$= \frac{\mu\gamma}{\lambda+2\mu} \frac{(t_2-t_1)r_2^3}{r_2^3 - r_1^3} (r_2 - r_1)(2r_2 + r_1) \ldots\ldots (13).*$$

The case when the thickness is small is interesting. Let $r_2 = r_1 + x$, then the maximum tension is

$$\frac{\mu\gamma(t_2-t_1)}{(\lambda+2\mu)} \left(1 + \frac{2x}{3r_1}\right),$$

neglecting the term $\frac{x}{r_1}$ in comparison with unity, we see that of two vessels the thicker is not sensibly more liable to break than the thinner, a result at first sight contradictory to experience. The explanation is that the greater liability of thick vessels to break is due to the fact that, allowing heat to pass through but slowly, a greater difference of temperature between the two surfaces really exists.

Let t'_2, t'_1 be the actual surface temperatures, we may assume that, if t_2 and t_1 be the temperatures of the surrounding media, the heat passing the two surfaces per unit of area will be $H_2(t_2 - t'_2)$ and $H_1(t'_1 - t_1)$.

Hence $$t'_2 - t'_1 = \frac{H_1 H_2 (t_2 - t_1)}{(H_1 + H_2)\frac{\kappa}{x} + H_1 H_2},$$

using this in the equation last obtained we have a result quite in accord with experience.

Returning now to equations (7) and (9), suppose the sphere to be solid and to be heated in any manner symmetrical about the centre. The constant b must vanish, and

$$\theta = \frac{\gamma}{\lambda+2\mu}\tau + 3a,$$

$$R = (\lambda+2\mu)\theta - 4\mu\frac{U}{r} - \gamma\tau$$

$$= -4\mu\frac{U}{r} + 3(\lambda+2\mu)a$$

$$= (3\lambda+2\mu)a - \frac{4\mu\gamma}{(\lambda+2\mu)r^3}\int r^2\tau dr;$$

* This result was set by me in the recent Mathematical Tripos Examination (Friday afternoon, January 17, 1879, Question ix).

now the mean temperature within the radius r is

$$\frac{4\pi \int r^2 \tau dr}{\frac{4}{3}\pi r^3} = \frac{3\int r^2 \tau dr}{r^3};$$

therefore, since the pressure is zero at the surface of the sphere,

$$R = \frac{4\mu\gamma}{3(\lambda+2\mu)} \cdot \{\text{mean temperature of whole sphere} - \text{mean temperature of sphere of radius } r\} \ldots (14),$$

$$T = \tfrac{1}{2}\frac{1}{r}\frac{dr^2 R}{dr} \ldots\ldots\ldots\ldots\ldots (14),$$

$$= \frac{4\mu\gamma}{3(\lambda+2\mu)} \cdot \{\text{mean temperature of whole sphere} - \tfrac{3}{2}\tau + \tfrac{1}{2}\text{mean temperature within the sphere of radius } r\} \ldots (15),$$

Other problems of the same character as the preceding will suggest themselves, for example that of a cylinder heated symmetrically around an axis, but as no present use could be made of the results I do not discuss them.

VORTEX MOTION IN A VISCOUS INCOMPRESSIBLE FLUID.

By *J. J. Thomson*, Trinity College, Cambridge.

Let u, v, w be the velocities of a particle of the fluid along the axes of x, y, z respectively; V the potential of the applied forces; p the pressure of the fluid at any point.

Then if the usual law of fluid friction be assumed, viz. that the tangential forces called into play by the viscosity are proportional to the relative velocities of two neighbouring elements of the fluid, the equations of motion are

$$\rho\left\{\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right\} - \rho\frac{dV}{dx} + \frac{dp}{dx} - \mu\nabla^2 u = 0 \ldots (1),$$

$$\rho\left\{\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right\} - \rho\frac{dV}{dy} + \frac{dp}{dy} - \mu\nabla^2 v = 0 \ldots (2),$$

$$\rho\left\{\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right\} - \rho\frac{dV}{dz} + \frac{dp}{dz} - \mu\nabla^2 w = 0 \ldots (3),$$

where μ is the coefficient of viscosity, and ρ the density of the fluid at any point.

Differentiating (1) with respect to y, (2) with respect to x, and subtracting, we get, if

$$\xi = \frac{dw}{dy} - \frac{dv}{dz},$$

$$\eta = \frac{du}{dz} - \frac{dw}{dx},$$

$$\zeta = \frac{dv}{dx} - \frac{du}{dy},$$

$$\frac{d\zeta}{dt} + \frac{du}{dx}\zeta + \frac{dv}{dy}\zeta - \xi\frac{dw}{dx} - \eta\frac{dw}{dy} + u\frac{d\zeta}{dx} + v\frac{d\zeta}{dy} + w\frac{d\zeta}{dz} = \frac{\mu\nabla^2}{\rho}\zeta,$$

but since the fluid is incompressible

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0;$$

and since

$$\frac{d\zeta}{dt} + u\frac{d\zeta}{dx} + v\frac{d\zeta}{dy} + w\frac{d\zeta}{dz} = \frac{D\zeta}{Dt},$$

where $\frac{D}{Dt}$ has its usual meaning, the equation reduces to

$$\frac{D\zeta}{Dt} - \xi\frac{dw}{dx} - \eta\frac{dw}{dy} - \zeta\frac{dw}{dz} = \frac{\mu}{\rho}\nabla^2\zeta,$$

similarly

$$\frac{D\eta}{Dt} - \xi\frac{dv}{dx} - \eta\frac{dv}{dy} - \zeta\frac{dv}{dz} = \frac{\mu}{\rho}\nabla^2\eta,$$

$$\frac{D\xi}{Dt} - \xi\frac{du}{dx} - \eta\frac{du}{dy} - \zeta\frac{du}{dz} = \frac{\mu}{\rho}\nabla^2\xi,$$

these are the equations of vortex motion in a viscous fluid and but for the term on the right-hand side are exactly the same as for a perfect fluid.

For small motions these equations reduce to

$$\frac{d\zeta}{dt} = \frac{\mu}{\rho}\nabla^2\zeta, \text{ \&c.}$$

These equations are the same as those which occur in conduction of heat. Hence we see by analogy that if we have a quantity of viscous fluid acted on by conservative forces, and if at any time the motion be irrotational then any vortex motion that may be in the fluid at any future time must have come from the boundary of the fluid.

We also see that if vortex motion be set up in any part of a viscous fluid the motion throughout the fluid immediately becomes rotational.

If the medium in which the vortex motion takes place, which, according to Sir W. Thomson's theory, constitutes matter, were viscous the qualities of bodies would not be permanent, the viscosity would tend to make all bodies possess the same properties.

Motion in two dimensions.

Let us consider the case in which every particle on the fluid moves in a plane parallel to the plane of xy, and let us suppose that initially there was an indefinitely long straight vortex filament whose axis coincided with the axis of z, in this case

$$\xi = 0, \quad \eta = 0, \quad w = 0,$$

and the equation

$$\rho \frac{D\zeta}{Dt} - \xi \frac{dw}{dx} - \eta \frac{dw}{dy} - \zeta \frac{dw}{dz} = \mu \nabla^2 \zeta$$

becomes

$$\rho \frac{D\zeta}{Dt} = \mu \nabla^2 \zeta,$$

since each particle of fluid describes a circle round the axis of z in a plane parallel to that of xy and since the motion is symmetrical round the axis of z

$$\frac{D\zeta}{Dt} = \frac{d\zeta}{dt},$$

and

$$\nabla^2 \zeta = \frac{d^2\zeta}{dr^2} + \frac{1}{r} \frac{d\zeta}{dr},$$

where r is the distance of the particle from the axis of z.

The equation of vortex motion thus becomes

$$\frac{d\zeta}{dt} = \frac{\mu}{\rho} \left(\frac{d^2\zeta}{dr^2} + \frac{1}{r} \frac{d\zeta}{dr} \right),$$

of which the solution is

$$\zeta = \zeta_0 \frac{\exp\left(-\frac{r^2}{4kt}\right)}{2\pi kt},$$

(Fourier, *Theorie de la chaleur*, Chap. IX.) where ρ_0 = strength of the original vortex filament, and $\exp u$ denotes e^u; also k is written in place of $\frac{\mu}{\rho}$.

To find the velocity at any point.

Let s be the velocity at a point whose polar coordinates are r, θ, since the point describes a circle of radius r,

$$u = -s\sin\theta, \quad v = s\cos\theta.$$

Now
$$\zeta = \frac{dv}{dx} - \frac{du}{dy}$$
$$= \frac{dv}{dr}\cos\theta - \frac{dv}{d\theta}\frac{\sin\theta}{r} - \frac{du}{dr}\sin\theta - \frac{du}{d\theta}\frac{\cos\theta}{r}$$
$$= \frac{ds}{dr} + \frac{s}{r}$$
$$= \frac{1}{r}\frac{d}{dr}(sr);$$

therefore
$$\frac{d}{dr}(sr) = r\frac{\exp\left(-\frac{r^2}{4kt}\right)}{2\pi kt}\zeta_0;$$

therefore
$$sr = -\zeta_0\frac{\exp\left(-\frac{r^2}{4kt}\right)}{\pi} + C,$$

when $t = 0$,
$$s = \frac{\zeta_0}{\pi r};$$

therefore
$$s = \frac{\zeta_0}{\pi r}\left\{1 - \exp\left(-\frac{r^2}{4kt}\right)\right\}.$$

We see that except when $t = 0$ the velocity is zero at the axis of z, a result which forms a remarkable contrast to the case of a perfect fluid when the velocity is infinite at the axis of z. The discontinuity which exists is a consequence of our initial circumstances being discontinuous, which obliged us to take a discontinuous function as the solution of the equations of motion.

To find the rate of diminution of the kinetic energy.
The rate of diminution of the kinetic energy

$$\propto -\int_0^\infty s\frac{ds}{dt}2\pi r\rho dr,$$

where s is the velocity,

$$\propto -\int_0^\infty \frac{\zeta_0}{\pi r}\left\{1-\exp\left(-\frac{r^2}{4kt}\right)\right\}\left\{-\frac{\zeta_0}{\pi r}\exp\left(-\frac{r^2}{4kt}\right)\right\}\frac{r^2}{4kt^2}2\pi r\rho dr,$$

$$\propto -\frac{\zeta_0^2\rho}{\pi.4kt^2}\int_0^\infty \left\{\exp\left(-\frac{r^2}{4kt}\right)-\exp\left(-\frac{2r^2}{4kt}\right)\right\}2r\,dr,$$

$$\propto -\frac{\zeta_0^2\rho}{\pi.2t},$$

if neither μ nor $t=0$; hence, we see that so long as μ is not 0 the rate of diminution of the kinetic energy does not depend on μ.

The vortex motion at a point distant r from the axis of z is greatest, when

$$\frac{d\zeta}{dt}=0,$$

i.e. when

$$\frac{\exp\left(-\frac{r^2}{4kt}\right)}{2\pi kt}\frac{r^2}{4kt^2}-\frac{\exp\left(-\frac{r^2}{4kt}\right)}{2\pi kt^2}=0$$

i.e. when

$$t=\frac{r^2}{4k},$$

i.e. the time which must elapse before the vortex motion at a point attains its maximum varies as the square of the distance of the point from the axis.

Suppose we have initially a single circular vortex in an infinite mass of fluid; to find the distribution of vortex motion, the motion being so small that squares of the velocity may be neglected.

Take the axis of the filament as the axis of z, let r be the distance of a point from the axis of z, θ the angle measured in a plane perpendicular to the axis of z, which this distance makes with a fixed line.

Let, as before,

$$\frac{\mu}{\rho}=k.$$

The equations of motion are

$$\frac{d\xi}{dt} = k\nabla^2\xi \quad \text{.....(1)},$$

$$\frac{d\eta}{dt} = k\nabla^2\eta \quad \text{..... .....(2)},$$

$$\zeta = 0.$$

Since the motion is symmetrical round the air of z, we may put

$$\xi = -s\sin\theta, \quad \eta = s\cos\theta,$$

where s is independent of θ.

Making these substitutions and transforming from x, y, z to r, θ, z, equations 1 and 2 reduce to

$$\frac{ds}{dt} = k\left(\frac{d^2s}{dz^2} + \frac{d^2s}{dr^2} + \frac{1}{r}\frac{ds}{dr} - \frac{s}{r^2}\right),$$

we may put

$$\frac{ds}{dt} = k\frac{d^2s}{dz^2} - hs \quad \text{.....} \quad (3),$$

then

$$\frac{d^2s}{dr^2} + \frac{1}{r}\frac{ds}{dr} + s\left(\frac{h}{k} - \frac{1}{r^2}\right) = 0 \quad \text{.....} \quad (4),$$

where h is any constant, since the solution of (3) with the conditions that, when $t = 0$, $s = 0$ except when $z = 0$ is

$$s = e^{-ht}\frac{\exp\left(-\frac{z^2}{4kt}\right)}{\sqrt{(2\pi kt)}} \text{ (function of } r),$$

and since the solution of 4 is

$$s = J_1\left(r\sqrt{\frac{h}{k}}\right) \text{ (function of } z \text{ and } t),$$

where J_1 is Bessel's function of the first order, we see

$$s = \Sigma Ce^{-ht}\frac{\exp\left(-\frac{z^2}{4kt}\right)}{\sqrt{(2\pi kt)}} J_1\left(r\sqrt{\frac{h}{k}}\right) \quad \text{.....(5)},$$

summation being taken for all positive values of h, since

$$\int_0^R J_1(r, a)\, J_1(r, b)\, r dr$$

$$= \frac{R}{b^2 - a^2}\left\{J_1(Ra)\frac{d}{dR}J(Rb) - J_1(Rb)\frac{d}{dR}J(Ra)\right\},$$

and since for large values of R

$$J_1(R) = \frac{\sqrt{2}}{\sqrt{(\pi R)}};$$

we see $$\int^{\infty} J_1(ra)\, J_1(rb)\, rdr = 0;$$

hence if we multiply each side of equation (5) by

$$rJ_1\left(r\sqrt{\frac{h}{k}}\right);$$

and integrate from 0 to ∞ we get if a be the radius of the original vortex filament

$$s = c\Sigma e^{-kt} \frac{\exp\left(-\frac{z^2}{4kt}\right)}{\sqrt{(2\pi kt)}} \frac{J_1\left(r\sqrt{\frac{h}{k}}\right) J_1\left(a\sqrt{\frac{h}{k}}\right)}{\int_0^{\infty} rJ_1^2\left(r\sqrt{\frac{h}{k}}\right) dr},$$

$c = s_0 a dr$, where s_0 is the value of s for the initial vortex; let $\frac{h}{k} = l^2$. Now

$$\int^R J_1^2(rl)\, rdr = R^2\left(\frac{d\,J_1(Rl)}{d\,(Rl)}\right)^2 + R^2\left(1 - \frac{1}{l^2R^2}\right) J_1^2(Rl),$$

if R be very great, $= \frac{2R}{\pi l}$, since then

$$J_1(R) = \sqrt{\left(\frac{2}{\pi R}\right)},$$

when R is very great the difference of successive roots of $J_1(R) = 0$, $= \pi$, hence $dl = \frac{\pi}{R}$; hence, changing from summation to integration

$$s = c\frac{\exp\left(-\frac{z^2}{4kt}\right)}{\sqrt{(2\pi kt)}} \int_0^{\infty} \exp(-kl^2t)\, J_1(rl)\, J_1(al)\, ldl.$$

Now $$\int_0^{\infty} \exp(-kl^2t)\, ldl = \tfrac{1}{2}\frac{1}{kt};$$

therefore

$$\int_0^\infty \exp(-kl^2t)\, l^{2n+1} dl = \tfrac{1}{2}\left\{\frac{d}{d(-kt)}\right\}^n \frac{1}{kt},$$

let

$$D = \frac{d}{d(-kt)},$$

then

$$s = c\,\frac{\exp\left(-\frac{z^2}{4kt}\right)}{\sqrt{(2\pi kt)}}\,\tfrac{1}{2}\,J_1\{r\sqrt{(D)}\}\,J_1\{a\sqrt{(D)}\}\,\frac{1}{kt}.$$

Since

$$J_1(r\sqrt{D}) = \frac{r\sqrt{D}}{2.2}\left(1 - \frac{rD}{2.4} + \frac{r^4D^2}{2.4.4.6} + \ldots \frac{(-)^n r^{2n} D^n}{2^{2n}.n!\,(n+1)!} + \ldots\right),$$

we have

$$J_1(r\sqrt{D})\,J_1(a\sqrt{D})$$

$$= \frac{ar\,D}{16}\left\{1 - \frac{(r^2+a^2)\,D}{2.4} + \frac{1}{2^4}\left(\frac{r^4}{2!\,3!} + \frac{a^4}{2!\,3!} + \frac{2a^2r^2}{2!\,2!}\right)D^2 + \ldots\right.$$

$$\left. + (-)^n \frac{1}{2^{2n}}\left(\frac{r^{2n}}{n!\,(n+1)!} + \frac{r^{2n-2}}{(n-1)!\,n!\,2!} + \frac{r^{2n-4}a^4}{(n-1)!\,(n-2)!\,2!\,3!} + \ldots\right)D^n\right\};$$

hence

$$s = c\,\frac{\exp\left(-\frac{z^2}{4kt}\right)}{\sqrt{(2\pi kt)}}\,\frac{ar}{32}\left\{\frac{1}{(kt)^2} - \frac{r^2+a^2}{2.4}\,\frac{2}{(kt)^3}\right.$$

$$\left. + \frac{1}{2^4}\left(\frac{r^4}{2!\,3!} + \frac{a^4}{2!\,3!} + \frac{a^2r^2}{2!\,2!}\right)\frac{2.3}{(kt)^4} + \ldots\right\}.$$

We see that there is no vortex motion at the axis of z, and that the distribution of vortex motion is similar in all planes perpendicular to this axis.

Since k and t only enter through the product kt, if equal vortex rings were initially in two fluids A and B, then the distribution of vortex motion in the fluid A at the time t would be the same as the distribution in the fluid B at the time $\frac{ka}{kb}t$, where ka and kb are the values of k for the fluids A and B respectively.

NOTES ON DETERMINANTS.

By *R. F. Scott, M.A.*

§1. Let

$$D_n = \begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & c, & a, & 0, & 0 \\ 1, & b, & c, & a, & 0 \\ 1, & 0, & b, & c, & a \\ 1, & 0, & 0, & b, & c \end{vmatrix} \quad (n \text{ rows}).$$

Where, with the exception of the border, the elements in the principal diagonal are equal to c, in the lines on either side of it a and b, the rest are zero.

By expanding with respect to the elements of the last row and column we may establish the following equation of differences for D_n,

$$D_n - cD_{n-1} + abD_{n-2} = \frac{(-a)^{n-1} + (-b)^{n-1}}{a+b+c} - \frac{c}{a+b+c} \cdot \frac{u^{n-1} - v^{n-1}}{u-v} + \frac{2ab}{a+b+c} \cdot \frac{u^{n-2} - v^{n-2}}{u-v},$$

where u and v are the roots of the equation

$$z^2 - cz + ab = 0.$$

Solving this we obtain

$$D_n = \frac{u^n + v^n}{(a+b+c)^2} - \frac{nc\,(u^n + v^n)}{(a+b+c)\,(u-v)^2} + \frac{2abn}{a+b+c} \frac{u^{n-1} + v^{n-1}}{(u-v)^2} - \frac{(-a)^n + (-b)^n}{(a+b+c)^2}.$$

§2. In establishing the equation of differences for D_n it was necessary to evaluate the determinant

$$\delta_n = \begin{vmatrix} 1, & 1, & 1, & 1, & 1 \\ c, & a, & 0, & 0, & 0 \\ b, & c, & a, & 0, & 0 \\ 0, & b, & c, & a, & 0 \\ 0, & 0, & b, & c, & a \end{vmatrix} \quad (n \text{ rows}),$$

and it is readily proved that

$$\delta_n = \frac{a^n}{a+b+c} + \frac{(-1)^{n-1}}{u-v}\,\frac{b(u^n-v^n)+u^{n+1}-v^{n+1}}{a+b+c}.$$

§3. It is known that

$$\begin{vmatrix} \alpha_1^n, \ldots \alpha_1^{i+1}, \alpha_1^{i-1}, \ldots \alpha_1, 1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_n^n, \ldots \alpha_n^{i+1}, \alpha_n^{i-1}, \ldots \alpha_n, 1 \end{vmatrix} = p_{n-i}\zeta^1(\alpha_1, \alpha_2, \ldots \alpha_n),$$

where $\zeta^1(\alpha_1, \alpha_2, \ldots \alpha_n)$ denotes as usual the product of

$$\begin{aligned} &(\alpha_1-\alpha_2), (\alpha_1-\alpha_3), \ldots(\alpha_1 \quad -\alpha_n), \\ &\qquad (\alpha_2-\alpha_3), \ldots(\alpha_2 \quad -\alpha_n), \\ &\qquad\qquad \ldots\ldots\ldots\ldots \\ &\qquad\qquad\qquad (\alpha_{n-1}-\alpha_n), \end{aligned}$$

and p_{n-i} is the sum of the products $n-i$ at a time without repetition of the quantities $\alpha_1, \alpha_2, \ldots \alpha_n$.

The simplest way to prove this seems to be to equate the coefficients of x^i in the identity

$$\begin{vmatrix} x^n, x^{n-1}, \ldots x, 1 \\ \alpha_1^n, \alpha_1^{n-1}, \ldots \alpha_1, 1 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_n^n, \alpha_n^{n-1}, \ldots \alpha_n, 1 \end{vmatrix} = \zeta^1(\alpha_1, \alpha_2, \ldots \alpha_n)(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n).$$

I do not think it has hitherto been noticed that this theorem may be extended. If in the array

$$\begin{matrix} \alpha_1^{n+r-1}, \alpha_1^{n+r-2}, \ldots \alpha_1, 1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_n^{n+r-1}, \alpha_n^{n+r-2}, \ldots \alpha_n, 1, \end{matrix}$$

we erase the r columns which contain the u^{th}, v^{th}, w^{th}, ... powers of the letters, and denote the determinant of the remaining columns by $D_{u,v,w,\ldots}$ we can shew that

$$D_{u,v,w,\ldots} = \begin{vmatrix} p_{n-u+r-1}, p_{n-u+r-2}, \cdots p_{n-u} \\ p_{n-v+r-1}, p_{n-v+r-2}, \cdots p_{n-v} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{vmatrix} \zeta^1(\alpha_1, \alpha_2, \ldots \alpha_n),$$

by forming an identity as above and equating coefficients.

We might also evaluate the determinant $D_{u,v,w,\ldots}$ by the method given by Cayley, Salmon, *Higher Algebra*, p. 290. But the above method has the advantage in this instance of giving a determinant on the right of order r instead of n, and in applications r is generally less than n.

§4. By substituting the exponential values for the sines and cosines, we can show by aid of the first theorem in §3 that for $2n$ angles

$$\begin{vmatrix} \sin a_1 \ldots \sin a_{2n} \\ \cos a_1 \ldots \cos a_{2n} \\ \sin 2a_1 \ldots \sin 2a_{2n} \\ \cos 2a_1 \ldots \cos 2a_{2n} \\ \ldots\ldots\ldots\ldots \\ \sin na_1 \ldots \sin na_{2n} \\ \cos na_1 \ldots \cos na_{2n} \end{vmatrix} = 2^{2n^2-2n+1} \Pi \sin \tfrac{1}{2}(a_i - a_k)\, S, \quad (i > k).$$

Where $S = \Sigma \cos \frac{1}{2}(a_1 + a_2 + \ldots + a_n - a_{n+1} - \ldots - a_{2n})$ is formed by dividing the $2n$ angles into two sets of n in all possible ways, and taking the cosine of half the difference of the sums of these two sets. Thus S is a sum of $\frac{1}{2}\frac{(2n)!}{n!\,n!}$ terms.

§5. By aid of the theorems in §3 and the theorem for expanding the determinant formed by compounding two rectangular arrays, we can shew that:

$$\begin{vmatrix} (\alpha-\alpha')^{n-1}, & (\alpha-\beta')^{n-1}, & \ldots (\alpha-\lambda')^{n-1} \\ (\beta-\alpha')^{n-1}, & (\beta-\beta')^{n-1}, & \ldots (\beta-\lambda')^{n-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (\lambda-\alpha')^{n-1}, & (\lambda-\beta')^{n-1}, & \ldots (\lambda-\lambda')^{n-1} \end{vmatrix} = C_{n-1}\zeta^{\frac{1}{2}}(\alpha, \beta, \ldots\lambda) \times \zeta^{\frac{1}{2}}(\alpha', \beta', \ldots\lambda').$$

Where here and in what follows $\alpha, \beta, \ldots\lambda$; $\alpha', \beta', \ldots\lambda'$ are any two sets of n quantities, and C_r is the product of all the coefficients in the expansion of $(1+x)^r$.

$$\begin{vmatrix} (\alpha-\alpha')^n, & \ldots (\alpha-\lambda')^n \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ (\lambda-\alpha')^n, & \ldots (\lambda-\lambda')^n \end{vmatrix} = \frac{C_n}{n!}\,\zeta^{\frac{1}{2}}(\alpha, \beta, \ldots\lambda)\,\zeta^{\frac{1}{2}}(\alpha', \beta', \ldots\lambda')\, I.$$

Where

$$I = \begin{Vmatrix} \alpha-\alpha', & \alpha-\beta', & \ldots \alpha-\lambda' \\ \beta-\alpha', & \beta-\beta', & \ldots \beta-\lambda' \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \lambda-\alpha', & \lambda-\beta', & \ldots \lambda-\lambda' \end{Vmatrix}$$

is a function formed like a determinant, only all the signs are + instead of ±. (I take this notation from a "Smith's Prize" paper, Jan. 30, 1873, set by Prof. Cayley).

If $u=(x-\alpha y)(x-\beta y)\ldots(x-\lambda y)$,
$v=(x-\alpha' y)(x-\beta' y)\ldots(x-\lambda' y)$,
$I=(12)^n uv$.

Using the notation of invariants, Salmon, *Higher Algebra*, § 151. In like manner we obtain the covariant determinant:

$$\begin{vmatrix} (\alpha-\alpha')^n, \ldots(\alpha-\lambda')^n, (\alpha-x)^n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (\lambda-\alpha')^n, \ldots(\lambda-\lambda')^n, (\lambda-x)^n \\ (x-\alpha')^n, \ldots(x-\lambda')^n, \end{vmatrix} = (-1)^n C_n \zeta^{\frac{1}{2}}(\alpha, \beta, \ldots\lambda) \times \zeta^{\frac{1}{2}}(\alpha', \beta', \ldots\lambda')\, uv.$$

$$\begin{vmatrix} (\alpha-\alpha')^{n+1}, \ldots(\alpha-\lambda')^{n+1}, (\alpha-x)^{n+1} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (\lambda-\alpha')^{n+1}, \ldots(\lambda-\lambda')^{n+1}, (\lambda-x)^{n+1} \\ (x-\alpha')^{n+1}, \ldots(x-\lambda')^{n+1}, \end{vmatrix} = (-1)^{n+1} \frac{C_{n+1}}{(n+1)!} \times \zeta^{\frac{1}{2}}(\alpha, \beta, \ldots\lambda) \times \zeta^{\frac{1}{2}}(\alpha', \beta', \ldots\lambda')\, uvI.$$

Where

$$I = \begin{Vmatrix} \alpha-\alpha', \alpha-\beta', \ldots\alpha-\lambda', \alpha-x \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \lambda-\alpha', \lambda-\beta', \ldots\lambda-\lambda', \lambda-x \\ x-\alpha', x-\beta', \ldots x-\lambda' \end{Vmatrix}$$
$$= -(12)^{n-1} uv.$$

§ 6. By slightly varying the form of the arrays we compound, we get other theorems of the same kind, *e.g.*

$$\begin{vmatrix} (\alpha-\alpha')^n, \ldots(\alpha-\lambda')^n, 1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (\lambda-\alpha')^n, \ldots(\lambda-\lambda')^n, 1 \\ 1 \quad \ldots \quad 1 \end{vmatrix} = (-1)^{n+1} C_n \zeta^{\frac{1}{2}}(\alpha, \beta, \ldots\lambda) \zeta^{\frac{1}{2}}(\alpha', \beta', \ldots\lambda').$$

$$\begin{vmatrix} (\alpha-\alpha')^{n+1}, \ldots(\alpha-\lambda')^{n+1}, 1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (\lambda-\alpha')^{n+1}, \ldots(\lambda-\lambda')^{n+1}, 1 \\ 1 \quad \ldots \quad 1 \end{vmatrix} = (-1)^n \frac{C_{n+1}}{(n+1)!} \zeta^{\frac{1}{2}}(\alpha, \beta, \ldots\lambda) \times \zeta^{\frac{1}{2}}(\alpha', \beta', \ldots\lambda')\, I'.$$

Where

$$I' = \begin{Vmatrix} \alpha-\alpha', \ldots\alpha-\lambda', 1 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \lambda-\alpha', \ldots\lambda-\lambda', 1 \\ 1 \quad \ldots \quad 1 \end{Vmatrix}$$
$$= (12)^{n-1} \frac{du}{dx} \cdot \frac{dv}{dx}.$$

By allowing $\alpha, \beta, \gamma, \ldots\lambda$; $\alpha', \beta', \ldots\lambda'$ to coincide, we get invariant and covariant determinants for a single quantic.

§ 7. If $$I = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} v\varepsilon^{-u} dx_1, dx_2, dx_n,$$

where $$v = \Sigma l_{ik} x_i x_k, \quad u = \Sigma a_{ik} x_i x_k.$$

Then $$I = \frac{\pi^{\frac{1}{2}n}}{2D^{\frac{1}{2}}} S.$$

Where $D = |\, a_{ik} \,|$ is the discriminant of u and S is the sum of the n determinants obtained by substituting for each column of D in succession the corresponding column of the discriminant of v; u is supposed to be incapable of becoming negative.

This is readily proved by transforming u and v by a simultaneous linear transformation, each to the sum of n squares.

By substituting pa_{ij} for the coefficients of u, and then differentiating with respect to p, we get other integrals.

§ 8. If there are two sets of n quantities $a, b, \ldots l$; $\alpha, \beta, \ldots\lambda$, and if

$$d^r_{ik} = (a_i - \alpha_k)^r + (b_i - \beta_k)^r + \ldots + (l_i - \lambda_k)^r.$$

Then

$$\begin{vmatrix} d^r_{11}, \ldots d^r_{1^s} \\ \ldots\ldots\ldots \\ d^r_{n1}, \ldots d^r_{ns} \end{vmatrix} = 0 \text{ if } s = n(r-1) + 3,$$

and

$$\begin{vmatrix} d^r_{11}, \ldots d^r_{1s}, 1 \\ \ldots\ldots\ldots\ldots \\ d^r_{n1}, \ldots d^r_{ns}, 1 \\ 1 \quad \ldots \quad 1 \end{vmatrix} = 0 \text{ if } s = n(r-1) + 2.$$

These two theorems include most of the determinant relations in geometry. For example, if we have two groups of points, x, y, z, ξ, η, ζ, and if a_{rs} denote the ratio of the square of the line joining the r^{th} point of the first, and the s^{th} point of the second group, to the square of the parallel semi-diameter of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Since $$a_{rs} = \frac{(x_r - \xi_s)^2}{a^2} + \frac{(y_r - \eta_s)^2}{b^2} + \frac{(z_r - \zeta_s)^2}{c^2}.$$

We see that

$$\begin{vmatrix} a_{11}, \ldots a_{16} \\ \ldots\ldots\ldots \\ a_{61}, \ldots a_{66} \end{vmatrix} = 0,$$

a relation between the lines joining two groups of six points in space.

It is also easy to shew that

$$\begin{vmatrix} a_{11}, \ldots a_{15} \\ \ldots\ldots\ldots \\ a_{51}, \ldots a_{55} \end{vmatrix} = 0,$$

if the five points of one group lie on an ellipsoid

$$\frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2} + \frac{(z-r)^2}{c^2} = m^2,$$

similar and similarly situated to the given one.

If the ellipsoid become a sphere we can replace a_{rs} by the squares of the lines joining the points.

It may be remarked that such space relations connected with an ellipsoid are not really more general than those connected with a sphere; for they are what the relations in an ordinary space become when it is altered by a homogeneous pure strain, such that the sphere

$$x^2 + y^2 + z^2 = R^2$$

becomes the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

NOTE ON CONTINUANTS.

By Professor *J. J. Sylvester*.

To find the number of terms in the cumulant or continuant $(a_1, a_2, \ldots a_n)$, we may proceed as follows:

1°. There is the term $a_1, a_2, a_3, \ldots a_n$.

2°. The number of terms of the first order of degradation, *i.e.* obtained by leaving out any pair of consecutive elements, is $n-1$, say $u_{n,1}$.

3°. The number of terms of the second order of degradation obtained by leaving out any two pairs of such, *i.e.* by leaving

out the first and second and some other pair of those that follow the second, the second and third and a pair of those that follow the third, the third and fourth and a pair of those that follow the fourth and so on, is

$$u_{n-2,1} + u_{n-3,1} + u_{n-4,1} + \dots,$$

and, consequently,

$$(n-3) + (n-4) + (n-5) + \dots$$
$$= \frac{(n-2)(n-3)}{2} =, \text{ say, } u_{n,2}.$$

4°. The number of the third order of degradation is in like manner

$$u_{n-2,2} + u_{n-3,2} + u_{n-4,2} + \dots,$$

i. e.
$$= \frac{(n-4)(n-5)}{1.2} + \frac{(n-5)(n-6)}{1.2} + \dots$$
$$= \frac{(n-3)(n-4)(n-5)}{1.2.3},$$

and so in general

$$u_{n,r} = u_{n-2,r-1} + u_{n-3,r-1} + u_{n-4,r-1} + \dots$$
$$= \frac{(n-r)(n-r-1)\dots(n-2r-1)}{1.2\dots r}.$$

Hence, the total number is

$$1 + (n-1) + \frac{(n-2)(n-3)}{1.2} + \frac{(n-3)(n-4)(n-5)}{1.2.3} + \dots.$$

Verification. In general

$$(i\sin\theta + \cos\theta)^n - (i\sin\theta - \cos\theta)^n$$
$$= 2\cos\theta\{(i\sin\theta)^{n-1} + (n-2)(i\sin\theta)^{n-3} + \frac{(n-3)(n-4)}{2}(i\sin\theta)^{n-5} + \dots\},$$

for we know that

$$\cos\theta\{(\sin\theta)^{n-1} - (n-2)(\sin\theta)^{n-3} + \frac{(n-3)(n-4)}{2}(\sin\theta)^{n-5} - \&c.\}$$

$= \cos n\theta$, or $\sin n\theta$, according as n is odd or even.

Hence, putting

$$i\sin\theta + \cos\theta = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{5},$$

so that
$$i\sin\theta - \cos\theta = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{5},$$
$$2i\sin\theta = 1,$$

and $$\cos\theta = \sqrt{5},$$

$$1 + (n-1) + \frac{(n-2)(n-3)}{1.2} + \frac{(n-3)(n-4)(n-5)}{1.2.3} + \ldots = \frac{(\frac{1}{2}+\frac{1}{2}\sqrt{5})^{n+1} - (\frac{1}{2}-\frac{1}{2}\sqrt{5})^{n+1}}{\sqrt{5}}.$$

But because

$$(a_1, a_2, \ldots a_n) = a_n(a_1, a_2, \ldots a_{n-1}) + (a_1, a_2, \ldots a_{n-2}),$$

if u_n is the number of terms in $(a_1, a_2, \ldots a_n)$,

$$u_n = u_{n-1} + u_{n-2},$$

with the initial conditions

$$u_0 = 1, \quad u_1 = 1.$$

Solving this difference-equation, we shall obtain

$$u_n = \frac{1}{\sqrt{5}}\{(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5})^{n+1} - (\tfrac{1}{2}-\tfrac{1}{2}\sqrt{5})^{n+1}\},$$

agreeing with the preceding result.

Corollary 1. The value of the continuant of the n^{th} order $(x, x, \ldots x)$, is

$$x^n + (n-1)x^{n-2} + \frac{(n-2)(n-3)}{1.2}x^{n-4} + \ldots,$$

which admits also of the clumsy representation

$$[\{\tfrac{1}{2}x + \tfrac{1}{2}\sqrt{(x^2+4)}\}^{n+1} - \{\tfrac{1}{2}x - \tfrac{1}{2}\sqrt{(x^2+4)}\}^{n+1}] \div \sqrt{(x^2+4)}.$$

Corollary 2. The value of the pro-continuant of the n^{th} order $(2\cos\theta, 2\cos\theta, \ldots 2\cos\theta)$, is $\dfrac{\sin(n+1)\theta}{\sin\theta}$.

By the pro-continuant is to be understood what a continuant becomes, when in its representative determinant, the oblique line of negative units are all changed into positive units so that the matrix has two precisely similar bands of units one above and one below the diagonal line and in opposition with it.

Corollary 3. The integral of the partial-difference equation

$$u_{x+1,y} - u_{x,y} - u_{x-1,y-1} = 0,$$

limited by the conditions

$$u_{x,0} = 1, \quad u_{x+1,x+1} = 0,$$

is

$$u_{x,y} = \frac{\Pi(x-y)}{\Pi(x-2y)\,\Pi y}.$$

EQUATION OF THE WAVE-SURFACE IN ELLIPTIC COORDINATES.

By Professor *Cayley*.

THE equation of the wave-surface

$$\frac{ax^2}{x^2+y^2+z^2-a}+\frac{by^2}{x^2+y^2+z^2-b}+\frac{cz^2}{x^2+y^2+z^2}=0,$$

when transformed to coordinates p, q, r, such that

$$\frac{x^2}{-a+p}+\frac{y^2}{-b+p}+\frac{z^2}{-c+p}=1,$$

$$\frac{x^2}{-a+q}+\frac{y^2}{-b+q}+\frac{z^2}{-c+q}=1,$$

$$\frac{x^2}{-a+r}+\frac{y^2}{-b+r}+\frac{z^2}{-c+r}=1;$$

(that is to the elliptic coordinates belonging to the quadric surface $\frac{x^2}{-a}+\frac{y^2}{-b}+\frac{z^2}{-c}=1$), assumes the form

$$(q+r-a-b-c)(r+p-a-b-c)(p+q-a-b-c)=0,$$

(Senate House Problem, January 14, 1879).

In fact p, q, r are the roots of the equation

$$\frac{x^2}{-a+u}+\frac{y^2}{-b+u}+\frac{z^2}{-c+u}=1;$$

we have therefore

$$(u-p)(u-q)(u-r)=(u-a)(u-b)(u-c)$$
$$-x^2(u-b)(u-c)-y^2(u-c)(u-a)-z^2(u-a)(u-b);$$

whence, writing for shortness

$$A=a+b+c \quad , \quad P=p+q+r,$$
$$B=bc+ca+ab, \quad Q=qr+rp+pq,$$
$$C=abc \quad , \quad R=pqr,$$

we have

$$x^2+\quad y^2+\quad z^2=P-A,$$
$$(b+c)x^2+(c+a)y^2+(a+b)z^2=Q-B,$$
$$bcx^2+\quad cay^2+\quad abz^2=R-C,$$

and thence also

$$a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=B(P-A)-(R-C),$$
$$ax^2+\quad by^2+\quad cz^2=A(P-A)-(Q-B).$$

The equation of the wave-surface is

$$abc-\{a(b+c)x^2+b(c+a)y^2+c(a+b)z^2\}$$
$$+(x^2+y^2+z^2)(ax^2+by^2+cz^2)=0,$$

and by the formulæ just obtained this is

$$C-[B(P-A)-(R-C)]+(P-A)[A(P-A)-(Q-B)]=0,$$

that is $A^3-2A^2P+A(P^2+Q)-(PQ-R)=0,$

that is $\{A-(q+r)\}\{A-(r+p)\}\{A-(p+q)=0,$

or, substituting for A its value $a+b+c$, and reversing the sign of each factor, we have the formula in question.

It is easy to see that, taking a, b, c to be each positive, $(a>b>c)$, and assuming also $p>q>r$, we obtain the different real points of space by giving to these coordinates respectively the different real values from ∞ to a, a to b, and b to c respectively. Hence

	greatest,	least value, is
$q+r$,	$a+b$,	$a+c$,
$r+p$,	∞ ,	$a+c$,
$p+q$,	∞ ,	$a+b$,

so that $r+p, p+q$, may be either of them $=a+b+c$, but $q+r$ cannot be $=a+b+c$, that is $q+r=a+b+c$ does not belong to any real point on the wave surface. We can only have $r+p$ and $p+q$ each $=a+b+c$, if $p=a+c$, $q=r=b$, and these values belong as is easily shown to the nodes on the wave-surface; hence, the equations $r+p=a+b+c$ and $p+q=a+b+c$ being satisfied simultaneously only at the nodes of the surface, must belong to the two sheets respectively. And it can be shown that $p+r=a+b+c$ belongs to the external sheet, and $p+q=a+b+c$ belongs to the internal sheet. In fact, for the point $(0, 0, \sqrt{a})$, which is on the external sheet, we have $p=a+c$, $q=a$, $r=b$, and therefore $p+r=a+b+c$: for the point $(0, 0, \sqrt{b})$, which is on the internal sheet, either

$$(p=b+c, q=a, r=b) \text{ or } (p=a, q=b+c, r=c),$$

according as $b+c>a$ or $b+c>a$, but in each case

$$p+q=a+b+c.$$

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, February 13th, 1879, C. W. Merrifield, Esq., F.R.S., *President*, in the Chair. Sir. J. Cockle, F.R.S., was admitted into the Society. Mr. R. Hargreaves, Fellow of St. Johns College, Cambridge, and Prof. W. E. Story, Leipzig, of the Johns Hopkins University, Baltimore, were proposed for election.

Dr. Hirst, F.R.S., communicated a paper by M. Halphen "On the number of conics which satisfy five independent conditions". Sir J. Cockle gave a construction for making Magic Squares; Prof. Cayley pointed out that the construction had been given in Leybourn's *Mathematical Repository*. Messrs. Harley, Henrici, Roberts, Hart, and other gentlemen took part in a discussion upon the subject, and gave different constructions. Prof. Henrici spoke upon Frames. Prof. H. J. S. Smith, F.R.S., read two papers, "On a Modular Equation" and "On the formula for four Abelian functions answering to the formula for the four theta functions." Mr. J. J. Walker, M.A., communicated a "Quaternion proof of Minding's theorem."

Thursday, March 13th, 1879, C. W. Merrifield, Esq., F.R.S., *President*, in the Chair. Mr. J. D. H. Dickson, M.A., was admitted into the Society. Mr. R. Hargreaves, M.A., and Prof. W. E. Story, Ph.D., were elected Members. Mr. Donald McAlister, B.A., Fellow of St. John's College, Cambridge, was proposed for election.

Prof. Cayley, F.R.S., spoke briefly, but in high praise, of the late Prof. Clifford's work as a mathematician, instancing more particularly the papers "On the Canonical Form and Dissection of a Riemann's Surface," "On Mr. Spottiswoode's Contact Problems," and "The Classification of Loci." The Chairman, the Rev. A. Freeman, M.A., and Dr. Hirst, F.R.S., added a few remarks on the loss the Society and the mathematical world generally had sustained, and expressed the hope that steps would be taken to secure the publication, if desirable, of any mathematical papers Prof. Clifford might have left.

Dr. Hirst made a statement respecting the "De Morgan Memorial" Medal to be presented to the Society (to be awarded by the Society in such manner as shall hereafter be determined). It appeared that the Bust and the Die for the Medal had been executed by Mr. Woolner, and that after all claims had been made a small sum would be still required to complete the amount requisite for the Medal Fund. A subscription list was commenced at the meeting, and Mr. Tucker was empowered to receive the subscriptions of such members as should be desirous of contributing to the special fund. Copies of the Medal were exhibited at the meeting.

The following communications were made: "On Differential Equations, Total and Partial; and on a new soluble class of the first, and an exceptional case of the second," Sir J. Cockle, F.R.S.; "Discussion of two double series arising from the number of terms in determinants of certain forms," Mr. J. D. H. Dickson, M.A.; "Two Geometrical Notes relating to Surfaces of the Second Order," Prof. H. J. S. Smith, F.R.S.

R. TUCKER, M.A., *Hon. Sec.*

THE END OF VOL. VIII.

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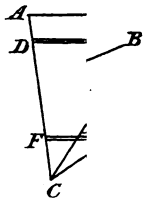
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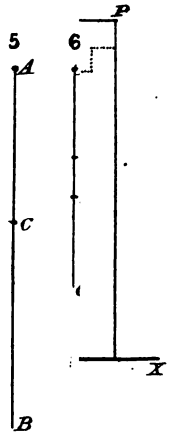
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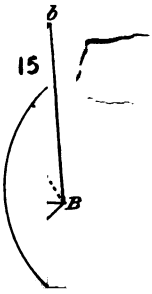
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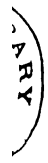

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